Binet - Cauchy Theorem

Let A be a rectangular matrix of size $m \times n$. Suppose $m \le n$ and J is an m-element subset of $\{1, 2, \dots, n\}$. Then we denote by $A[J]$ the matrix $(A_{i,j})_{\substack{i=1,\dots,m \\ j \in J}}$ consisting of columns of A indexed by elements of J.

Example.

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 5 & 6 & 7 \end{pmatrix}, \quad J = \{1, 3\}, \quad A[J] = \begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix}.$$

Theorem (Binet-Cauchy)

Let $A, B \in M_{m \times n}(\mathbb{C})$. If $m \le n$ then

$$\det(A \cdot B^t) = \sum_J \big(\det A[J]\big)\big(\det B[J]\big)$$

where J runs over all m-element subsets of $\{1, 2, \dots, n\}$.

Proof: We have

$$A = (A_{i,j})_{(i,j)\in[m]\times[n]}, \quad B = (B_{i,j})_{(i,j)\in[m]\times[n]} \text{ and}$$

$$(AB^T)_{i_1,i_2} = \sum_{j=1}^{n} A_{i_1,j}\cdot B_{i_2,j}.$$ We compute:

$$\det(AB^T) = \sum_{\pi\in S_m} \operatorname{sign}(\pi) \prod_{i=1}^{m} (AB^T)_{i,\pi(i)}$$

$$= \sum_{\pi\in S_m} \operatorname{sign}(\pi) \prod_{i=1}^{m} \left(\sum_{j=1}^{n} A_{i,j}\cdot B_{\pi(i),j}\right)$$

$$= \sum_{\pi\in S_m} \operatorname{sign}(\pi) \sum_{(j_1,\dots,j_m)\in\{1,\dots,n\}^m} \prod_{i=1}^{m} A_{i,j_i}\cdot B_{\pi(i),j_i}$$

$$\sum_{\substack{J\subset\{1,\dots,n\}\\ |J|=m}} \det A[J]\cdot \det B[J]$$

$$= \sum_{1\le j_1<j_2<\dots<j_m\le n} \left(\sum_{\pi_1\in S_m} \operatorname{sign}(\pi_1)\prod_{i=1}^{m} A_{i,j_{\pi_1(i)}}\right)\cdot \left(\sum_{\pi_2\in S_m} \operatorname{sign}(\pi_2)\prod_{i=1}^{m} B_{i,j_{\pi_2(i)}}\right)$$

Lemma: Let $(j_1,\dots,j_m)\in\{1,\dots,n\}^m$. Suppose that $j_s = j_t$ for some pair of indices $s\neq t$. Then

$$\star = \sum_{\pi\in S_m} \operatorname{sign}(\pi) \prod_{i=1}^{m} A_{i,j_i}\cdot B_{\pi(i),j_i} = 0.$$

Proof: Let $\sigma_{s,t} \in S_m$ be the permutation that interchanges s and t and fixes all other elements of $[m]$.

Then for each $\pi \in S_m$ we have

$$\operatorname{sign}(\pi \circ \sigma) = -\operatorname{sign}(\pi).$$

$$\prod_{i=1}^{m}\left(A_{i,j_i} \cdot B_{\pi\circ\sigma(i),j_i}\right) = \prod_{i \in [m]\setminus\{s,t\}}\left(A_{i,j_i} \cdot B_{\pi\circ\sigma(i),j_i}\right) \cdot \prod_{i \in \{s,t\}}\left(A_{i,j_i} \cdot B_{\pi\circ\sigma(i),j_i}\right)$$

$$= \prod_{i \in [m]\setminus\{s,t\}}\left(A_{i,j_i} \cdot B_{\pi(i),j_i}\right) \cdot A_{s,j_s} \cdot B_{\pi(t),j_s} \cdot A_{t,j_t} \cdot B_{\pi(s),j_t}$$

$$= \prod_{i \in [m]\setminus\{s,t\}}\left(A_{i,j_i} \cdot B_{\pi(i),j_i}\right) \cdot A_{s,j_s} \cdot B_{\pi(t),j_t} \cdot A_{t,j_t} \cdot B_{\pi(s),j_s}$$

$$= \prod_{i=1}^{m}\left(A_{i,j_i} \cdot B_{\pi\circ(i),j_i}\right).$$

Thus

$$\operatorname{sign}(\pi) \prod_{i=1}^{m} A_{i,j_i} \cdot B_{\pi(i),j_i} =$$

$$= - \operatorname{sign}(\pi \circ \sigma) \prod_{i=1}^{m} A_{i,j_i} \cdot B_{\pi \circ \sigma(i),j_i} .$$

Since the map $\pi \mapsto \pi \circ \sigma$ is a bijection on S_m

$$\star = \sum_{\pi \in S_m} \operatorname{sign}(\pi) \prod_{i=1}^{m} A_{i,j_i} \cdot B_{\pi(i),j_i} = - \star$$

Therefore $\star = 0$.

This finishes the proof of the claim. $\blacksquare$

$$\det(AB^T) = \sum_{\pi \in S_m} \operatorname{sign}(\pi) \prod_{i=1}^{m} (AB^T)_{i,\pi(i)}$$

$$= \sum_{\pi \in S_m} \operatorname{sign}(\pi) \prod_{i=1}^{m} \left(\sum_{j=1}^{n} A_{i,j} \cdot B_{\pi(i),j} \right)$$

$$= \sum_{\pi \in S_m} \operatorname{sign}(\pi) \sum_{(j_1,\dots,j_m) \in \{1,\dots,n\}^m} \prod_{i=1}^{m} A_{i,j_i} \cdot B_{\pi(i),j_i}$$

use the lemma

$$= \sum_{\pi \in S_m} \operatorname{sign}(\pi) \sum_{\substack{(j_1,\dots,j_m) \in \{1,\dots,n\}^m \\ \text{all coordinates are distinct}}} \prod_{i=1}^{m} A_{i,j_i} \cdot B_{\pi(i),j_i}$$

$$= \sum_{\pi \in S_m} \operatorname{sign}(\pi) \sum_{1 \le j_1 < \dots < j_m \le n} \sum_{\pi' \in S_m} \prod_{i=1}^{m} A_{i,j_{\pi'(i)}} \cdot B_{\pi(i),j_{\pi'(i)}}$$

$$\boxed{\begin{aligned} &\sum_{\substack{J \subset \{1,\dots,n\} \\ |J| = m}} \det A[J] \cdot \det B[J] \\ &= \sum_{1 \le j_1 < j_2 < \dots < j_m \le n} \left(\sum_{\pi_1 \in S_m} \operatorname{sign}(\pi_1) \prod_{i=1}^{m} A_{i,j_{\pi_1(i)}} \right) \cdot \left(\sum_{\pi_2 \in S_m} \operatorname{sign}(\pi_2) \prod_{i=1}^{m} B_{i,j_{\pi_2(i)}} \right) \end{aligned}}$$

$$\sum_{\pi \in S_m} \operatorname{sign}(\pi) \sum_{1 \leq j_1 < \dots < j_m \leq n} \sum_{\pi' \in S_m} \prod_{i=1}^{m} A_{i, j_{\pi'(i)}} \cdot B_{\pi(i), j_{\pi'(i)}}$$

$$= \sum_{1 \leq j_1 < \dots < j_m \leq n} \sum_{\pi \in S_m} \sum_{\pi' \in S_m} \operatorname{sign}(\pi) \left(\prod_{i=1}^{m} A_{i, j_{\pi'(i)}} \right) \cdot \left(\prod_{i=1}^{m} B_{i, j_{\pi' \circ \pi^{-1}(i)}} \right) \circledequal$$

We make a change of variables: $\pi' =: \pi_1$, $\pi' \circ \pi^{-1} =: \pi_2$

Then $\pi = \pi_2^{-1} \circ \pi_1$ and $\operatorname{sign}(\pi) = \operatorname{sign}(\pi_1) \cdot \operatorname{sign}(\pi_2)$.

$$\circledequal \sum_{1 \leq j_1 < \dots < j_m \leq n} \sum_{\pi_1 \in S_m} \sum_{\pi_2 \in S_m} \operatorname{sign}(\pi_1) \operatorname{sign}(\pi_2) \left(\prod_{i=1}^{m} A_{i, j_{\pi_1(i)}} \right) \cdot \left(\prod_{i=1}^{m} B_{i, j_{\pi_2(i)}} \right)$$

$$\sum_{\pi \in S_m} \operatorname{sign}(\pi) \sum_{1 \le j_1 < \dots < j_m \le n} \sum_{\pi' \in S_m} \prod_{i=1}^{m} A_{i, j_{\pi'(i)}} \cdot B_{\pi(i), j_{\pi'(i)}}$$

$$= \sum_{1 \le j_1 < \dots < j_m \le n} \sum_{\pi \in S_m} \sum_{\pi' \in S_m} \operatorname{sign}(\pi) \left(\prod_{i=1}^{m} A_{i, j_{\pi'(i)}} \right) \cdot \left(\prod_{i=1}^{m} B_{i, j_{\pi' \circ \pi^{-1}(i)}} \right) \overset{\circ}{=}$$

We make a change of variables: $\pi' =: \pi_1$, $\pi' \circ \pi^{-1} =: \pi_2$

Then $\pi = \pi_2^{-1} \circ \pi_1$ and $\operatorname{sign}(\pi) = \operatorname{sign}(\pi_1) \cdot \operatorname{sign}(\pi_2)$.

$$\overset{\circ}{=} \sum_{1 \le j_1 < \dots < j_m \le n} \sum_{\pi_1 \in S_m} \sum_{\pi_2 \in S_m} \operatorname{sign}(\pi_1) \operatorname{sign}(\pi_2) \left(\prod_{i=1}^{m} A_{i, j_{\pi_1(i)}} \right) \cdot \left(\prod_{i=1}^{m} B_{i, j_{\pi_2(i)}} \right)$$

$$= \sum_{1 \le j_1 < \dots < j_m \le n} \left(\sum_{\pi_1 \in S_m} \operatorname{sign}(\pi_1) \left(\prod_{i=1}^{m} A_{i, j_{\pi_1(i)}} \right) \right) \cdot \left(\sum_{\pi_2 \in S_m} \operatorname{sign}(\pi_2) \left(\prod_{i=1}^{m} B_{i, j_{\pi_2(i)}} \right) \right)$$

Finally, we observe that

$$\sum_{1 \leq j_1 < \ldots < j_m \leq n} \left(\sum_{\pi_1 \in S_m} \operatorname{sign}(\pi_1) \left(\prod_{i=1}^{m} A_{i, j_{\pi_1(i)}} \right) \right) \cdot \left(\sum_{\pi_2 \in S_m} \operatorname{sign}(\pi_2) \left(\prod_{i=1}^{m} B_{i, j_{\pi_2(i)}} \right) \right)$$

$$= \sum_{1 \leq j_1 < \ldots < j_m \leq n} \det \left(A[\{j_1, \ldots, j_m\}] \right) \cdot \det \left(B[\{j_1, \ldots, j_m\}] \right)$$

$$= \sum_{\substack{J \subseteq \{1, \ldots, n\} \\ |J| = m}} \det \left(A[J] \right) \cdot \det \left(B[J] \right).$$

This finishes the proof of Binet-Cauchy Theorem $\blacksquare$