Generating functions

Combinatorial applications of polynomials

Philosophical observation:

Natural numbers and finite sets are two sides of the same coin

Another observation:

Arithmetic operations and operations with finite sets have many similarities.

<u>Example:</u>

$$(a+b)\cdot c = a\cdot c + b\cdot c$$

$$(A\cup B)\cap C = (A\cap C)\cup(B\cap C)$$

algebra & analysis

Idea: Use the power of arithmetic manipulations to speed up combinatorial computations.

Combinatorial applications of polynomials

<u>Example</u>: How many ways are there to pay the amount of 21 franks with

6 one-francs coins
5 two-francs coins
4 five-francs coins ?

<u>Solution</u>: The number in question is the number of solutions of

$$x_1 + x_2 + x_3 = 21 \quad (\star)$$

$x_1 \in \{0, 1, 2, \ldots, 6\}$, $x_2 \in \{0\cdot 2, 1\cdot 2, \ldots, 5\cdot 2\}$, $x_3 \in \{0\cdot 5, 1\cdot 5, 2\cdot 5, 3\cdot 5, 4\cdot 5\}$

Consider the following polynomials

$$P_1(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6$$

$$P_2(x) = 1 + x^2 + x^4 + x^6 + x^8 + x^{10}$$

$$P_3(x) = 1 + x^5 + x^{10} + x^{15} + x^{20}$$

The number of solutions of $(\star)$ is the coefficient of x^{21} in the product $P_1(x)\cdot P_2(x)\cdot P_3(x)$.

Operations with polynomials

$$P(x) = \sum_{k=0}^{n} a_k x^k \qquad Q(x) = \sum_{k=0}^{n} b_k x^k$$

1. Addition

$$(P+Q)(x) = \sum_{k=1}^{n} (a_k + b_k) x^k$$

2. Multiplication

$$P(x) \cdot Q(x) = \sum_{k=1}^{2n} (a_0 b_k + a_1 b_{k-1} + \ldots + a_k b_0) x^k$$

3. Substitution

$$P(Q(x)) = a_0 + a_1 \cdot Q(x) + a_2 Q(x)^2 + \ldots + a_n Q(x)^n$$

4. Differentiation

$$\frac{d}{dx} P(x) = \sum_{k=1}^{n} a_k \cdot k \cdot x^{k-1} = \sum_{k=0}^{n-1} (k+1) a_{k+1} x^k$$

5. Integration

$$\int_0^x P(y)\,dy = \sum_{k=0}^{n} a_k \frac{x^{k+1}}{k+1} = \sum_{k=1}^{n+1} \frac{a_{k-1}}{k} \cdot x^k$$

Identities with binomial coefficients.

Example: Prove the identity

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

Solution. Consider the identity

$$(1+x)^n \cdot (1+x)^n = (1+x)^{2n} \qquad (\ast)$$

Now we compute the coefficient of x^n on the left-hand side and on the right-hand side of $(\ast)$

On the left:

$$(1+x)^n \cdot (1+x)^n = \left(\binom{n}{0} + \binom{n}{1} x + \dots + \binom{n}{n} x^n \right) \cdot \left(\binom{n}{0} + \binom{n}{1} x + \dots + \binom{n}{n} x^n \right)$$

coefficient of x^n:

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \dots + \binom{n}{n}\binom{n}{0} = \binom{n}{0}^2 + \binom{n}{1}^2 + \dots + \binom{n}{n}^2$$

on the right:

$$(1+x)^{2n} = \binom{2n}{0} + \binom{2n}{1} x + \dots + \binom{2n}{n} x^n + \dots + \binom{2n}{2n} x^{2n}$$

This finishes the proof.

One more identity with binomial coefficients.

Example: Compute $\sum_{k=0}^{n} k \cdot \binom{n}{k}$.

Solution:

Consider the polynomial $P(x) = (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

Differentiate $P(x)$:

$$\frac{d}{dx} P(x) = \frac{d}{dx}\left[(1+x)^n\right] = n(1+x)^{n-1}$$

On the other hand:

$$\frac{d}{dx} P(x) = \frac{d}{dx}\left[\sum_{k=0}^{n} \binom{n}{k} x^k\right] = \sum_{k=1}^{n} k \binom{n}{k} x^{k-1}$$

Thus:

$$\sum_{k=1}^{n} k \binom{n}{k} x^{k-1} = n(1+x)^{n-1}$$

Substitute $x = 1$: $\sum_{k=1}^{n} k \binom{n}{k} = n \cdot 2^{n-1}$ $\blacksquare$

Multinomial theorem.

Theorem: The following holds:

$$(x_1 + x_2 + \dots + x_n)^k = \sum_{\substack{i_1, i_2, \dots, i_n \geq 0 \\ i_1 + i_2 + \dots + i_n = k}} \frac{k!}{i_1! \, i_2! \cdots i_n!} \, x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$$

Proof: $\dfrac{k!}{i_1! \, i_2! \cdots i_n!}$ is the number of sequences of length k from letters „x_1", „x_2", …, „x_n".

such that letter „x_1" is used i_1-times
„x_2" is used i_2-times
⋮
„x_n" is used i_n-times.

We prove the theorem by induction by n.

Basis of induction: $n=2$ Binomial theorem.

Step of induction:

Suppose that $$(x_1+\dots+x_n)^k=\sum_{\substack{i_1,\dots,i_n\geq 0\\ i_1+\dots+i_n=k}}\frac{k!}{i_1!\,i_2!\cdots i_n!}x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$$

$$((x_1+\dots+x_n)+x_{n+1})^k=\sum_{i=0}^{k}\binom{k}{i}(x_1+\dots+x_n)^{k-i}x_{n+1}^{i}$$

$$=\sum_{i=0}^{k}\left(\sum_{\substack{i_1,\dots,i_n\geq 0\\ i_1+\dots+i_n=k-i}}\frac{(k-i)!}{i_1!\,i_2!\cdots i_n!}x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\right)\binom{k}{i}x_{n+1}^{i}$$

$i=i_{n+1}$

$$=\sum_{\substack{i_1,\dots,i_{n+1}\geq 0\\ i_1+\dots+i_{n+1}=k}}\frac{\cancel{(k-i_{n+1})!}}{i_1!\,i_2!\cdots i_n!}\cdot\frac{k!}{\cancel{(k-i_{n+1})!}\,i_{n+1}!}\cdot x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\cdot x_{n+1}^{i_{n+1}}$$

$\blacksquare$