Exercise Set Solutions #6
"Discrete Mathematics" (2025)

E1. Prove the following identities for Fibonacci numbers. In each identity below $n \geq 1$.

(a) $F_1 + F_3 + F_5 \ldots + F_{2n-1} = F_{2n}$.

(b) $F_{2n+1} = 3F_{2n-1} - F_{2n-3}$.

(c) $F_{a+b+1} = F_{a+1}F_{b+1} + F_aF_b$.

(d) $\gcd(F_n, F_{n+1}) = 1$.

> ***Solution:*** (a) Let $n = 1$. From the definition we know $F_2 = F_1 + F_0$ and $F_0 = 0$. Hence $F_2 = F_1$ and the (a) holds for $n = 1$. Let us assume (a) holds for $n - 1$. By definition of Fibonacci series $F_n = F_{n-1} + F_{n-2}$ and due to the assumption
>
> $$F_1 + F_3 + F_5 \ldots + F_{2n-3} = F_{2n-2}.$$
>
> Therefore,
>
> $$F_{2n} = F_{2n-1} + F_{2n-2} = F_{2n-1} + F_1 + F_3 + F_5 \ldots + F_{2n-3}.$$
>
> (b) By using $F_n = F_{n-1} + F_{n-2}$ we get
>
> $$\begin{aligned} F_{2n+1} &= F_{2n} + F_{2n-1} = 2F_{2n-1} + F_{2n-2} \\ &= 2F_{2n-1} + F_{2n-1} - F_{2n-3} = 3F_{2n-1} - F_{2n-3}. \end{aligned}$$
>
> (c) We will show $F_{n+m} = F_{n-1}F_m + F_nF_{m+1}$ by induction over m. Let $m = 1$ then $F_n = F_{n-1} + F_n = F_{n-1}F_1 + F_nF_2$ since $F_1 = F_2 = 1$. For $m = 2$, we have
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> $$F_{n-1}F_2 + F_nF_3 = F_{n-1}F_2 + F_nF_2 + F_nF_1 = F_{n+1}F_2 + F_nF_1 = F_{n+1} + F_n = F_{n+2}.$$
>
> Assume the statement holds up to $m - 1$. We show that it also holds for m
>
> $$\begin{aligned} F_{n+m} &= F_{n+m-1} + F_{n+m-2} \\ &= F_{n-1}F_{m-1} + F_nF_m + F_{n-1}F_{m-2} + F_nF_{m-1} \\ &= F_{n-1}(F_{m-1} + F_{m-2}) + F_n(F_m + F_{m-1}) \\ &= F_{n-1}F_m + F_nF_{m+1} \end{aligned}$$
>
> By setting $a = n - 1$ and $b = n$, we have $F_{a+b+1} = F_{a+1}F_{b+1} + F_aF_b$.
> (d) We do an induction on n. For $n = 1$, it is clear. In general, if $d = \gcd(F_n, F_{n+1}) = \gcd(F_n, F_n + F_{n-1})$, then d would have to divide both F_{n-1} and F_n, but $\gcd(F_{n-1}, F_n) = 1$, so $d = 1$.

E2. Prove that any positive integer can be written as a sum of mutually distinct Fibonacci numbers.

> ***Solution:*** In this solution we will call a decomposition of $m \in \mathbb{N}$ a set of mutually distinct Fibonacci numbers such that they sum up to m.
>
> Let $f(n)$ be the greatest Fibonacci number F_i such that $F_i \leq n$. We can easily check that 0 is the sum of 0 distinct Fibonacci numbers. Then if we have a decomposition for all $0 \leq m < n$, this means that we have a decomposition for $n - f(n)$ as $f(n)$ is strictly positive. Thus the

decomposition of n is the decomposition of $n - f(n)$ except we add $f(n)$ to it. This is a valid decomposition as $\forall n \in \mathbb{N}^* f(n) > \frac{n}{2}$ because Fibonacci numbers are a strictly increasing sequence after they leave 1 , and thus $f(n)$ cannot appear in the decomposition of $n - f(n)$ because $f(n) > n - f(n)$ and Fibonacci numbers are positive.

E3. Consider the sequence $(a_0, a_1, a_2 \ldots)$ with $a_0 = 1, a_1 = 2, a_2 = 3$ and

$$a_{k+1} = 5a_k - 8a_{k-1} + 4a_{k-2}$$

for $k \geq 2$. Find an expression for the value of a_k. What is its generating function?

Solution: Using the theorem for linear recurrences we want to solve the equation $x^3 = 5x^2 - 8x + 4$. This equation is equivalent to $(x-1)(x-2)^2$. Therefore we have the root 1 with multiplicity one and root 2 with multiplicity 2 . The coefficients in the sequence therefore are of the form $a_n = c_0(1)^n + (c_1 n + c_2)\, 2^n$. Using the initial conditions $a_0 = 1, a_1 = 2, a_2 = 3$ we have to solve the following equations to obtain the values for c_0, c_1 and c_2 :

$$\begin{aligned} 1 &= a_0 = c_0 + c_2 \\ 2 &= a_1 = c_0 + (c_1 + c_2) \cdot 2 \\ 3 &= a_2 = c_0 + (2c_1 + c_2) \cdot 2^2 \end{aligned}$$

This has the solution $c_0 = -1, c_1 = \frac{-1}{2}, c_3 = 2$. Therefore the values a_k have the form

$$a_k = -1 + \left(\frac{-k}{2} + 2\right) 2^k.$$

For the generating function $a(x)$ we get from the recurrence relation and initial conditions that

$$\begin{aligned} a(x) &= 1 + 2x + 3x^2 + 3x^3 - 1x^4 + -17x^5 \ldots + a_n x^n + \ldots \\ -5xa(x) &= 0 - 5x - 10x^2 - 15x^3 - 15x^4 + 5x^5 \ldots - 5a_{n-1}x^n + \ldots \\ 8x^2 a(x) &= 0 + 0x + 8x^2 + 16x^3 + 24x^4 + 24x^5 \ldots + 8a_{n-2}x^n + \ldots \\ -4x^3 a(x) &= 0 + 0x + 0x^2 - 4x^3 - 8x^4 - 12x^5 \ldots - 4a_{n-3}x^n + \ldots \end{aligned}$$

Hence $\left(1 - 5x + 8x^2 - 4x^3\right) a(x) = 1 - 3x + x^2$ and therefore $a(x) = \frac{1-3x+x^2}{1-5x+8x^2-4x^3}$.

E4. What is the generating function of the sequence $(a_0, a_1, a_2 \ldots)$ with $a_0 = 1, a_1 = 3$ and $a_k = 3a_{k-1} - 2a_{k-2}$ for $k \geq 2$?

Solution: Since

$$\begin{aligned} a(x) &= 1 + 3x + 7x^2 + 15x^3 + 31x^4 + \ldots + a_n x^n + \ldots \\ -3xa(x) &= 0 - 3x - 9x^2 - 21x^3 - 45x^4 + \ldots - 3a_{n-1}x^n + \ldots \\ 2x^2 a(x) &= 0 + 0x + 2x^2 + 6x^3 + 14x^4 + \ldots + 2a_{n-2}x^n + \ldots \end{aligned}$$

we have $\left(1 - 3x + 2x^2\right) a(x) = 1$. Therefore $a(x) = \frac{1}{1-3x+2x^2}$.

E5. Suppose that $a_0 = 2, a_1 = 8$ and for $n \geq 0$ we have $a_{n+2} = \sqrt{a_n a_{n+1}}$. Can you write an expression for the general a_n ? What is $\lim_{n\to\infty} a_n$?

know that $a_n > 0$, by induction. So we can consider $x_n = \log_2 a_n$. Then $x_0 = 1, x_1 = 1$ and for $n \geq 0$ we have $x_{n+2} = \frac{1}{2}(x_n + x_{n+1})$. The characteristic polynomial of this recurrence is $x^2 = \frac{1}{2}(x+1)$. But note that $2x^2 - x - 1 = 2x^2 - 2x + x - 1 = (x-1)(2x+1)$, and by adjusting the constants, we get

$$x_n = \frac{7}{3}1^n - \frac{4}{3}\left(-\frac{1}{2}\right)^n$$
$$\Rightarrow a_n = 2^{\frac{7}{3} - \frac{4}{3}\left(-\frac{1}{2}\right)^n}.$$

Finally, when $n \to \infty$ we get $\lim_{n\to\infty} a_n = 2^{\frac{7}{3}}$.

E6. Let $a(n,k) = \#\{A \subset [n] : |A| = k, A \text{ does not contain two consecutive elements}\}$.

(a) Prove that

$$a(n,k) = a(n-1,k) + a(n-2,k-1) \text{ for } k \geq 2$$

and use it to compute the generating functions $A_k(x) = \sum_{n\geq 1} a(n,k)x^n$.

Solution: We prove the recurrence equation using combinatorial method. For $A \subseteq [n]$ with $|A| = k$ with no two consecutive elements, if $n \notin A$ then $A \subseteq [n-1]$ and there is $a(n-1,k)$ ways of choosing A. If $n \in A$ then $n - 1 \notin A$ by hypothesis, $A \setminus \{n\} \subseteq [n-2]$, and thus there is $a(n-2,k-1)$ ways of choosing A. Of course, this only works if $k \geq 2$. We conclude that

$$a(n,k) = a(n-1,k) + a(n-2,k-1) \text{ for } k \geq 2.$$

Now, notice that the generating function of (a) satisfies for $k \geq 2$

$$A_k(x) = xA_k(x) + x^2 A_{k-1}(x)$$

which leads to

$$A_k(x) = A_{k-1}(x)\frac{x^2}{1-x}.$$

Hence, we get that for $k \geq 2$

$$A_k(x) = \left(\frac{x^2}{1-x}\right)^{k-1} A_1(x) = \left(\frac{x^2}{1-x}\right)^k \frac{1}{x(1-x)},$$

where

$$A_1(x) = \sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2} \text{ and } A_0(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

(b) Use item (a) to prove that

$$\sum_{k\geq 0} \binom{n-k+1}{k} = F_{n+2}$$

Solution: First, it is easy to note that $a_{n,k} = \binom{n-k+1}{k}$. Thus, it is enough to prove that the generating function of the sequence $n \mapsto F_{n+2}$ is

$$\begin{aligned}
\sum_{k\geq 0} A_k(x) &= \frac{1}{x(1-x)} \sum_{k\geq 1} \left(\frac{x^2}{1-x}\right)^k + \frac{1}{1-x} \\
&= \frac{1}{x(1-x)} \left(\sum_{k\geq 0} \left(\frac{x^2}{1-x}\right)^k - 1 \right) + \frac{1}{1-x} \\
&= \frac{1}{x(1-x)} \left(\frac{1}{1-\frac{x^2}{1-x}} - 1 \right) + \frac{1}{1-x} \\
&= \frac{1+x}{1-x-x^2}.
\end{aligned}$$

In fact, if we denote $F(x)$ the generating function of the Fibonnaci sequence, we get that the generating function of $n \mapsto F_{n+2}$ is

$$\begin{aligned}
\sum_{n\geq 0} x^n F_{n+2} &= \frac{1}{x^2} \sum_{n\geq 0} x^{n+2} F_{n+2} \\
&= \frac{1}{x^2} \left(F(x) - xF_1 - x^0 F_0 \right) \\
&= \frac{1}{x^2} \left(\frac{x}{1-x-x^2} - x \right) \\
&= \frac{1+x}{1-x-x^2}
\end{aligned}$$

concluding.