Exercise Set Solutions #12

"Discrete Mathematics" (2025)

E1. Recall the definition of the Ramsey numbers: We define $R(k,l)$ to be the minimum number of vertices n required to guarantee that any graph with n vertices has a clique of size k or an independent set of size l. Prove that $k \leq k'$ and $l \leq l'$ implies $R(k,l) \leq R(k',l')$.

> ***Solution:*** We know that for a graph with $R(k',l')$ vertices we can find a clique of size k' or an independent set of size l' by the definition of the Ramsey numbers. Since $k \leq k'$ and $l \leq l'$ we can also find a clique of size k or an independent set of size l. Therefore $R(k,l) \leq R(k',l')$.

E2. **(a)** Show that the Ramsey number $R(3,3)$ is 6.

(b) Let G be a graph with at least $\binom{k+l-2}{k-1}$ vertices. Show that there is a clique of size k or an independent set of size l.

> ***Solution:***
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> **(a)** For 5 vertices we can construct a graph that neither have a clique of size 3 nor an independent set of size 3 , for example a pentagon. To prove that every graph G with 6 vertices has a clique of size 3 or an independent set of size 3 we do the following: Color all edges of G blue and complete the graph with red edges. Note that now a clique of red edges corresponds to an independent set in the original graph. Now pick a vertex v. There are 5 edges incident to v. By the pigeonhole principle at least 3 of them must have the same colour. Without loss of generality we consider the 3 edges to be blue (if not we can switch the colors). We assume they connect v to the vertices u, w and r. If any of the edges $(uw), (ur), (rw)$ exists, we found a clique of size 3 . If none of these edges exist we found the independent set u, w, r of size 3 . In either case this proves the claim.
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> **(b)** This is Ramseys theorem. We show the finitenes of the Ramsey numbers. We want to adapt the above idea and proceed by induction on $k+l$. We therefore color the existing edges blue again and complete the graph by adding red edges. For $k=1$ or $l=1$ the claim holds: We know that $\binom{k+1-2}{k-1} = \binom{1+l-2}{0} = 1$ and every graph with at least one vertex has a clique and an independent set of size 1 . Now let $k, l \geq 2$ and consider a graph with $n = \binom{k+l-2}{k-1}$ vertices. We know by the induction hypothesis that the statement holds for $k, l-1$ and for $k-1, l$ in graphs with $n_1 = \binom{k+l-3}{k-1}$ and respectively $n_2 = \binom{k+l-3}{k-2}$ vertices. We know from Pascals triangle that $n = n_1 + n_2$. We again choose a random vertex v. It has n edges incident to it. By the pigeon hole principle we have that either the number of red edges incident to it is $\geq n_1$ or the number of blue edges is $\geq n_2$. If we have at least n_1 red edges incident to it, we use the induction hypothesis for $(k, l-1)$. By the induction hypothesis we have either a blue clique of size k (in this case we are done) or a red clique of size $l-1$. In this case we can extend the clique with v and obtain a red clique of size l. For the second case of having at least n_2 blue edges incident to v we proceed analogously. This finishes the proof.

E3. Let $|X| = n$ with $n \geq 2k$. Show that the Erdos-Ko-Rado theorem is sharp, i.e. that there exists an intersecting family $\mathcal{F}$ of k-element subsets of X such that

$$|\mathcal{F}| = \binom{n-1}{k-1}$$

> ***Solution:*** We do this by constructing such a family. Let us fix one point $a \in X$. We choose the family $\mathcal{F}$ to be all $(k-1)$-element subsets of $X \setminus \{a\}$ and adjoin a to each of these sets. We then obtain subsets of the size k and since all of our subsets contain a the fmaily is intersecting.

E4. Prove that each tournament has a Hamiltonian path.

> ***Solution:*** We proceed by induction on the number n of vertices. For $n = 2$ the statement is true. Assume the claim holds for every tournament with $n-1$ vertices. Consider now a tournament T with n vertices. We choose a random vertex v and group the remaining vertices into two sets. We call the vertices with edges going to v tournament T_1 and the vertices with edges coming from v tournament T_2. Now T_1 and T_2 have at most $n-1$ vertices since v belongs to neither of them. Therefore by the induction hypothesis there exists a Hamiltonian path in T_1 and T_2. Now take a path in T_1, continue it to v (possible by our choice of T_1) and continue from v to a path in T_2. This gives us a path in T.

E5. Prove that in any tournament there exists a vertex v that can be reached from any other vertex by a directed path of length at most 2.

> ***Solution:*** We proceed by induction on the number n of vertices. For $n = 2$ the statement is true. Assume the claim holds for every tournament with $n-1$ vertices. Consider now a tournament T with n vertices. We delete one vertex u and let v be the vertex from the induction hypothesis for the tournament $T\backslash\{u\}$. If in the tournament T the vertex v can be reached from vertex u in two steps, this proves the claim. If v can not be reached from u in two steps, that means that the edge (v, u) from v to u is in the tournament and also all vertices with an arrow going to v have an arrow going to u. Therefore u can be reached from everywhere within two steps. This proves the claim.

E6. Let K_n denote the complete graph on n vertices. Suppose one colours all the edges of K_n with one of two colours: red or blue.

(a) Let $v \in V(K_n)$ be a vertex. A bad cherry with vertex v is a set of three vertices $u, v, w \in V(K_n)$ such that the colour of the edge uv is different from the colour of the edge vw.
If $r(v)$ denotes the number of edges coming out of vertex v which are painted red, show that the number of bad cherries with vertex v is exactly $r(v)(n-1-r(v))$.

(b) We say that three vertices $u, v, w \in V(K_n)$ form a monochromatic triangle if the edges uv, uw, vw all have the same colour. Show that, for any colouring of the edges of K_n as above, the number of monochromatic triangles is at least

$$\frac{1}{4}\binom{n}{3} - n^2$$

> ***Solution:***
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> **(a)** For creating a bad cherry one would need to pick a red edge coming out of v (for which there are $r(v)$ options) and one blue edge coming out of v (for which there are $n-1-r(v)$ options). Thus, we conclude that the number of bad cherries is exactly $r(v)(n-1-r(v))$.
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> **(b)** We count the number of bad triangles instead. Notice that for each bad triangle there are

2 bad cherries. Thus, the number of bad triangles is

$$\#\text{ bad }\Delta's = \frac{1}{2}\sum_{v\in V} r(v)(n-1-r(v)).$$

The function $x \to x(n-1-x)$ is maximized in $x = (n-1)/2$, so

$$\#\text{ bad }\Delta's \leq \frac{1}{2}\sum_{v\in V}(n-1)^2/4 = n(n-1)^2/8 = \frac{3}{4}\binom{n}{3} + \frac{n(n-1)}{8} \leq \frac{3}{4}\binom{n}{3} + n^2.$$

As there are $\binom{n}{3}$ possible triangles, we get that the number of monochromatic triangles is at least:

$$\geq \binom{n}{3} - \left(\frac{3}{4}\binom{n}{3} + n^2\right) = \frac{1}{4}\binom{n}{3} - n^2$$