Exercise Set Solutions #11

"Discrete Mathematics" (2025)

Exercise 7 is to be submitted on Moodle before 23:59 on May 12th, 2025

E1. In a kindergarden, there are 12 boys, 3 of whom are 3 years old, 5 are 4 years old and 4 are 5 years old; and 9 girls, 4 of whom are 3 years old, 2 are 4 years old and 3 are 5 years old. We pick one child, each with equal probability.

(1) What is the probability of picking a girl?

(2) What is the probability of picking a girl, provided that we pick a 3 years-old?

(3) What is the probability of picking a 3 years-old, provided that it is a girl?

Answer the same questions for boys.

> ***Solution:***
>
> **(a)** The probability is $3/7$.
>
> **(b)** The probability is $4/7$.
>
> **(c)** The probability is $4/9$.

E2. Let $\mathcal{F}$ be a family of 3 -element subsets of a finite set X. Prove that the elements of X can be colored with 3 colors so that at least $|\mathcal{F}|3!/3^3$ sets in $\mathcal{F}$ have exactly one element of each color.

> ***Solution:*** Choose a random 3-coloring of the elements of X so that each element gets one of the 3 colors independently with probability $1/3$, and let C denote the random variable that counts the number of sets in $\mathcal{F}$ that have exactly one element of each color. We have
>
> $$C = \sum_{Y \in \mathcal{F}} I_Y$$
>
> where I_Y denotes the indicator random variable which is 1 if Y has exactly one element of each color and 0 otherwise, for $Y \in \mathcal{F}$. We have
>
> $$\mathbb{E}[I_Y] = P(I_Y = 1) = 3!/3^3$$
>
> because there are 3^3 colorings of the elements of Y and 3 ! of them assign each color to exactly one element of Y. By the linearity of expectation, we have
>
> $$\mathbb{E}[C] = \mathbb{E}\left[\sum_{Y \in \mathcal{F}} I_Y\right] = \sum_{Y \in \mathcal{F}} \mathbb{E}[I_Y] = \sum_{Y \in \mathcal{F}} \frac{3!}{3^3} = |\mathcal{F}| \cdot 3!/3^3$$
>
> It follows that there is a coloring for which $C \geq |\mathcal{F}| \cdot 3!/3^3$, that is, at least $|\mathcal{F}| \cdot 3!/3^3$ sets in $\mathcal{F}$ have exactly one element of each color.

E3. Compute the expected number of 3-cycles in a random graph on n vertices. Here the probability space is the set of all possible graphs (of which there are $2^{\binom{n}{2}}$ of those) and each random graph is assumed to be equally likely.

Solution: Let G be a random graph. Let the vertices be $[n]$. Let $I_{ij}(G)$ denote the random variable which is 1 if $\{i, j\} \in E(G)$ and 0 otherwise (both these values appear with a probability of 0.5).

Now we know that for any three vertex set $\{i, j, k\} \subseteq [n], i, j, k$ distinct, they form a cycle if and only if

$$I_{ij}(G) I_{jk}(G) I_{ki}(G)=1$$

It is clear that

$$\mathbb{E}\left(I_{ij}(G) I_{jk}(G) I_{ki}(G)\right)=P\left(I_{ij}(G) I_{jk}(G) I_{ki}(G)\right)=1)=\frac{1}{8}.$$

This can be shown either directly, or through independence of random variables. As a result, we conclude that the expected number of cycles is

$$\mathbb{E}\left(\sum_{\{i, j, k\} \subseteq[n]} I_{ij}(G) I_{jk}(G) I_{ki}(G)\right)=\frac{\binom{n}{3}}{8}.$$

E4. Let $\{v_1, \ldots, v_n\}$ be unit vectors in $\mathbb{R}^d$. Prove that it is possible to choose signs $\varepsilon_i \in\{\pm 1\}$ such that the vector $\sum_{i=1}^n \varepsilon_i v_i$ has Euclidean norm less than or equal to $\sqrt{n}$.

Solution: Let $X_{\varepsilon_1, \ldots, \varepsilon_n}=\left\|\sum_{i=1}^n \varepsilon_i v_i\right\|$. We choose the weights $\varepsilon_1, \ldots, \varepsilon_n$ independently and uniformly at random, and for convenience, we consider the square of the Euclidean norm. By the linearity of expectation, we obtain that

$$\begin{aligned}
\mathbb{E}\left[X_{\varepsilon_1, \ldots, \varepsilon_n}^2\right] & =\mathbb{E}\left[\left\|\sum_{i=1}^n \varepsilon_i v_i\right\|^2\right]=\mathbb{E}\left[\left\langle\sum_{i=1}^n \varepsilon_i v_i, \sum_{i=1}^n \varepsilon_i v_i\right\rangle\right]=\mathbb{E}\left[\sum_{i=1}^n \varepsilon_i^2\left\|v_i\right\|^2+\sum_{i, j=1, i \neq j}^n \varepsilon_i \varepsilon_j\left\langle v_i, v_j\right\rangle\right] \\
& =\mathbb{E}\left[n+\sum_{i, j=1, i \neq j}^n \varepsilon_i \varepsilon_j\left\langle v_i, v_j\right\rangle\right]=n+\mathbb{E}\left[\sum_{i, j=1, i \neq j}^n \varepsilon_i \varepsilon_j\left\langle v_i, v_j\right\rangle\right]
\end{aligned}$$

The expected value of the last sum is zero. Indeed, since ε_i and ε_j are independent, we have

$$\mathbb{E}\left[\sum_{i, j=1, i \neq j}^n \varepsilon_i \varepsilon_j\left\langle v_i, v_j\right\rangle\right]=\sum_{i, j=1, i \neq j}^n \mathbb{E}\left[\varepsilon_i\right] \mathbb{E}\left[\varepsilon_j\right]\left\langle v_i, v_j\right\rangle=\sum_{i, j=1, i \neq j}^n 0 \cdot\left\langle v_i, v_j\right\rangle=0.$$

In conclusion, the expected value of the square of the norm is n, so there is at least one choice of the weights for which the vector has norm at least $\sqrt{n}$.

E5. (1) For a graph $G=(V, E)$, we denote the complement of G as $G'=\left(V,\binom{V}{2} \backslash E\right)$. That is v_1, v_2 is an edge in G' if and only if $\{v_1, v_2\} \notin E$. We call a set $S \subset V(G)$ a clique if for two $s_1, s_2 \in S,\{s_1, s_2\} \in E(G)$. Then apply Turán's theorem on G' to prove the following equivalent version.

If G has n vertices but no cliques of size $r+1$, then

$$|E| \leq \frac{r-1}{r}\frac{n^2}{2}.$$

(2) For any given value $s, t \in \mathbb{Z}_{\geq 1}$, find a graph G_t on $n = s \cdot t$ vertices with $s \cdot t \cdot (t-1)/2$ edges such that $\alpha(G_t) = s$. Check that this is equal to the lower bound on independence number in Turán's theorem for each t.

Solution:

(1) Observe that $S \subset V$ is a clique in G if and only if it is an independent set in G'. Hence, if G is has no cliques of size $r+1$, then G' has no independent sets of size $r+1$. Thus, an independent set in G' has to be of size r at most. By Turán's theorem, we are guaranteed an independent set of size at least $n^2/\left(2\left|E\left(G'\right)\right|+1\right)$. So we write

$$r \geq \frac{n^2}{\left(2\binom{n}{2} - |E|\right) + n} \Rightarrow |E| \leq \frac{r-1}{r}\frac{n^2}{2}.$$

(2) Take the graph G_t to be the disjoint union of s complete graphs K_t. Then, it has the given number of vertices. Turán's inequality is then

$$\alpha\left(G_t\right) \geq \frac{(st)^2}{st(t-1)+st} = s$$

On the other hand, it is clear that any subset of size strictly bigger than s will have to contain at least two points inside one of the copies of K_t and hence cannot be independent. Therefore, we are assured that $\alpha\left(G_t\right) = s$.

The complement of this graph will be a complete multipartite graph. There will be s sets of t vertices such that each copy of t vertices don't have any internal edges but any two vertices in different copies will have an edge in between. You can check that this G_t' tightly satisfies the equivalent version above.

E6. Find $\alpha(G)$ when G is one of the following graphs. Compare it with the lower bound given by Turán's theorem.

(1) The complete graph $K_n = \left([n], \binom{[n]}{2}\right)$. That is to say that K_n is a graph with n vertices such that there is an edge between any two vertices.

(2) The complete bipartite graph $K_{n,m} = ([n] \sqcup [m], [n] \times [m])$. That is, the vertices are into two groups of size n and m and there is an edge between each vertex of one group to another.

(3) A path graph $P_n = ([n], E)$ where $E = \{\{i, i+1\}\}_{i=1}^{n-1}$.

(4) A circular graph $C_n = P_n + \{1, n\}$.

Solution:

(1) Any two vertices in K_n are connected by an edge. Hence, it is impossible to find two independent vertices. Therefore, $\alpha\left(K_n\right) = 1$.

Turán's theorem would give us

$$\alpha\left(K_n\right) \geq \frac{n^2}{n(n-1)+n} = 1$$

(2) Any two vertices lying on opposite sides of the bipartite graph are connected. Hence, an independent set should completely be on one side. Maximizing this, we see that $\alpha\left(K_{m,n}\right) = \max\{m, n\}$. Turán's theorem gives us now

$$\alpha\left(K_{n,m}\right) \geq \frac{(n+m)^2}{2nm+n+m}.$$

(3) Starting from any edge, we can pick up vertices alternatingly. This gives us $\alpha\left(P_n\right) = \lfloor n/2 \rfloor + 1$. Using Turán's theorem tells us

$$\alpha\left(P_n\right) \geq \frac{n^2}{2(n-1)+n} = \frac{n}{3-\frac{2}{n}}$$

(4) Again, we are forced to pick up edges alternatingly. With this, we get $\alpha\left(C_n\right) = \lfloor n/2 \rfloor$. On the other hand, from Turán's theorem we get

$$\alpha\left(C_n\right) \geq \frac{n^2}{2n+n} = \frac{n}{3}$$