Exercise Set Solutions #10

"Discrete Mathematics" (2025)

Note: Spanning trees exist only for connected graphs. All graphs in this exercise set are to be assumed as connected.

E1. Let G be a graph and $L(G)$ be the Laplace matrix. If $L_0(G)$ is the matrix obtained by removing the last row and column of $L(G)$, show that

$$\det L_0(G) = \frac{1}{n}\lambda_1\lambda_2\dots\lambda_{n-1}$$

where $\lambda_1, \lambda_2, \dots, \lambda_{n-1}$ are the eigenvalues of $L(G)$ with multiplicities (could be possibly zero).
Hint: $\det\left(L(G) - xI_n\right) = -x\left(\lambda_1 - x\right)\left(\lambda_2 - x\right)\dots\left(\lambda_{n-1} - x\right)$.

> ***Solution:*** In $L(G)$, the sum of all the rows and columns is 0 . Adding all the columns to the last one, and then all the rows to the last one, we get
>
> $$\left|\begin{array}{ccc|c} & & L_0 - xI_{n-1} & \begin{matrix} L_{1n} \\ L_{2n} \\ \vdots \end{matrix} \\ \hline L_{1n} & L_{2n} & \cdots & L_{nn} - x \end{array}\right| = \left|\begin{array}{ccc|c} & & L_0 - xI_{n-1} & \begin{matrix} -x \\ -x \\ \vdots \end{matrix} \\ \hline L_{1n} & L_{2n} & \cdots & -x \end{array}\right|$$
>
> $$\left|\begin{array}{ccc|c} & & L_0 - xI_{n-1} & \begin{matrix} -x \\ -x \\ \vdots \end{matrix} \\ \hline -x & -x & \cdots & -nx \end{array}\right| = -x \left|\begin{array}{ccc|c} & & L_0 - xI_{n-1} & \begin{matrix} -x \\ -x \\ \vdots \end{matrix} \\ \hline 1 & 1 & \cdots & n \end{array}\right|$$
>
> Now to get the coefficient of $-x$ in $\det\left(L(G) - xI_n\right)$, we have to look at the constant term in the following determinant,
>
> $$\left|\begin{array}{ccc|c} & & L_0 - xI_{n-1} & \begin{matrix} -x \\ -x \\ \vdots \end{matrix} \\ \hline 1 & 1 & \cdots & n \end{array}\right|.$$
>
> To get that, we put $x = 0$ in the expression, and get that $n\det\left(L_0\right) = \lambda_1\lambda_2\dots\lambda_{n-1}$.

E2. Find the number of spanning trees of the following graphs.

(1) The complete graph $K_n = \left([n], \binom{[n]}{2}\right)$. That is to say that K_n is a graph with n vertices such that there is an edge between any two vertices.

(2) The complete bipartite graph $K_{n,m} = ([n] \sqcup [m], [n] \times [m])$. That is, the vertices are into two groups of size n and m and there is an edge between each vertex of one group to another.

(3) A path graph $P_n = ([n], E)$ where $E = \{\{i, i+1\}\}_{i=1}^{n-1}$.

(4) A circular graph $C_n = P_n + \{1, n\}$.

Solution: (1) This is actually just the number of trees on n vertices. It is equal to n^{n-2} by Cayley's formula. But we can also prove this by finding the value of the determinant of the following $(n-1)\times(n-1)$ matrix and dividing by n.

$$\begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{bmatrix}$$

(2) We can actually find out the eigenvalues of the Laplace matrix directly. The Laplace matrix $L(G)$ is the following $(n+m)\times(n+m)$ matrix.

$$\left[\begin{array}{ccc|ccc} & & & -1 & \ldots & -1 \\ & mI_n & & \vdots & \ddots & \vdots \\ & & & -1 & \ldots & -1 \\ \hline -1 & \ldots & -1 & & & \\ \vdots & \ddots & \vdots & & nI_m & \\ -1 & \ldots & -1 & & & \end{array}\right]$$

Any column vector with 0 in the bottom m entries and such that the sum of all entries is 0 , is an eigenvector of $L(G)$ with eigenvalue m. Similarly, if the top n entries are 0 and the total sum is 0 , it is an eigenvector with eigenvalue n. Hence, the eigenspace of eigenvalue m is of dimension $n-1$ and eigenspace of n has dimension $m-1$.

Finally one last eigenvector one can make is by taking m in the top n entries and $-n$ in the last m entries. This corresponds to an eigenvalue $(m+n)$. Multiplying them together and dividing by $m+n$ gives

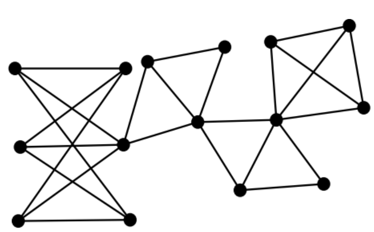
$$\frac{1}{m+n}\left[(m+n)m^{n-1}n^{m-1}\right]=m^{n-1}n^{m-1}$$

(3) No need to use the Kirchhoff's theorem here! The graph itself is already a tree. It has exactly 1 spanning subtree.
(4) Deleting and edge from the circular graph gives us a tree and any subtree must be of this form. Hence, there are n possible spanning trees.

E3. Suppose G_1 and G_2 are graphs with exactly 1 vertex in common. That is $V(G_1)\cap V(G_2)=\{v\}$ for some v. If $T(G)$ denotes the number of spanning trees of a graph G, then show that

$$T(G_1\cup G_2)=T(G_1)\,T(G_2)$$

Here $V(G_1\cup G_2)=V(G_1)\cup V(G_2)$ and $E(G_1\cup G_2)=E(G_1)\cup E(G_2)$. Now find $T(G)$ where G is the following graph



Solution: Let $S(G)$ be the set of spanning trees of a graph G. Then we want to show the following bijection.

$$\begin{gathered} S(G_1) \times S(G_2) \rightarrow S(G_1 \cup G_2) \\ (T_1, T_2) \mapsto T_1 \cup T_2 \end{gathered}$$

Indeed, the assignment is bijective since T_1 and T_2 can be uniquely recovered from a spanning tree of $G_1 \cup G_2$. Hence the map is both surjective and injective.

For the second part, we observe that G can be written as a union of $K_{3,3}$, two copies of a diamond shaped graph D and K_4. In the class, we saw that $T(D) = 8$. From the previous question, we know that $T(K_{3,3}) = 3^2 3^2 = 81$ and $T(K_4) = 4^2 = 16$. So the answer is $T(G) = 81 \times 8 \times 8 \times 16 = 82944$

E4. Prove the following lemma: Suppose $S \subseteq [m]$. Then,

(1) For $\sigma \in \operatorname{Perm}([m])$ such that $[m] \backslash S \subseteq [m]^\sigma$, we have that $\left.\sigma\right|_S \in \operatorname{Perm}(S)$. Here $\left.\sigma\right|_S : S \rightarrow [m]$ is the restriction of σ to S and $[m]^\sigma$ denotes the set of fixed points under the permutation σ.

(2) For σ as above, we get $\operatorname{sgn}(\left.\sigma\right|_S) = \operatorname{sgn}(\sigma)$.

(3) The mapping $\sigma \mapsto \left.\sigma\right|_S$ is a bijection between

$$\{\sigma \in \operatorname{Perm}([m]) \mid [m] \backslash S \subseteq [m]^\sigma\} \leftrightarrow \operatorname{Perm}(S).$$

Solution: (1) If everything outside S is fixed by $\sigma : [m] \rightarrow [m]$, it is clear that $\left.\sigma\right|_S$ will take values only in S. Indeed, if $x \in [m] \backslash S$ is such that $\sigma(y) = x$, then $y = x$ and hence $y \in [m] \backslash S$. Now $\left.\sigma\right|_S : S \rightarrow S$ is an injective function, and S is a finite set so it must be a bijection.

(2) Since, $\left.\sigma\right|_S$ is a permutation of S, we can find $\operatorname{sgn}(\left.\sigma\right|_S)$ by finding whether or not it can be composed using an odd number of transpositions or even. For any decomposition of $\left.\sigma\right|_S = \tau_1 \tau_2 \ldots \tau_r$ in terms of transpositions $\tau_i \subseteq \operatorname{Perm}(S)$, we can decompose $\sigma = \tilde{\tau}_1 \tilde{\tau}_2 \ldots \tilde{\tau}_r$ where $\tilde{\tau}_i \in \operatorname{Perm}([m])$ is the transposition τ_i lifted to $\operatorname{Perm}([m])$ by just fixing everything in $[m] \backslash S$. Hence, if r is odd, both $\operatorname{sgn}(\sigma)$ and $\sigma(\left.\sigma\right|_S)$ are -1 , and similarly they are both +1 when r is even.

(3) The inverse map is to send a permutation $\tau \in \operatorname{Perm}(S)$ to a permutation $\tilde{\tau} \in \operatorname{Perm}([m])$ which is

$$\tilde{\tau}(i) = \begin{cases} \tau(i) & i \in S \\ i & i \notin S \end{cases}$$

The reader is welcome to check that this is an inverse map and the two sets are in bijection.

E5. Recall from linear algebra the notion of the adjoint matrix. For a matrix A, we define matrix $\operatorname{adj}(A)$ as

$$\operatorname{adj}(A)_{ji} = (-1)^{i+j} \det A^{(i,j)}$$

where $A^{(i,j)}$ is the matrix obtained by A after deletion of i th row and j th column. Now let G be a graph and $L(G)$ be the Laplace matrix of G

(1) Let $v \in \mathbb{C}^n$ be the column vector with all 1s. Verify that $L(G)v = 0$.

(2) What is $\operatorname{rank}(L(G))$? What is then the null space of $L(G)$?

(3) What is $\det(L(G))$? What is the product $L(G) \times \operatorname{adj}(L(G))$?

(4) Conclude that $\left|\det L(G)^{(i,j)}\right|$ is the indepenent of i, j. Use this to further conclude that the proof of Kirchhoff's theorem given in the class did not depend on which row was removed from the incidence matrix.

Solution: (1) This is just saying that the sum of rows of $L(G)$ is zero.
(2) We know that $\det L_0(G) = \det L(G)^{(n,n)}$ is non-zero from Kirchhoff's theorem since it counts the number of spanning trees of G. Hence $\operatorname{rank}(L_0(G)) \geq n-1$. But since we have at least one non-zero vector in the null-space, namely the vector v, we get $\operatorname{rank}(L_0(G)) = n-1$.

This implies that the null-space is the one-dimensional subspace generated by v. That is $\ker L_0(G) = \mathbb{C}v$.
(3) $\det L(G) = 0$ because it is not a full-rank matrix. We know that $L(G) \times \operatorname{adj}(L(G)) = \det(L(G))I_n = 0$.
(4) Each column of adj $L(G)$ lies in the null-space of $L(G)$. So we conclude that each column of adj $L(G)$ is equal to Cv for some $C \in \mathbb{C}$. Since $L(G)$ is symmetric, adj $L(G)$ is also symmetric and therefore each entry of adj $L(G)$ is equal to C for some $C \in \mathbb{C}$.

If we were to delete the i th row from the incidence matrix $M(G, \mathcal{O})$ and call that matrix $M^{(i)}(G, \mathcal{O})$, then it can be verified that $M^{(i)}(G, \mathcal{O})M^{(i)}(G, \mathcal{O})^t = L(G)^{(i,i)}$. The determinant of this matrix is the same as that of $L_0(G)$.

E6. Let $T(K_n)$ be the number of spanning trees of the complete graph K_n, as defined above. Show that

$$(n-1)T(K_n) = \sum_{k=1}^{n-1} k(n-k)\binom{n-1}{k-1} T(K_k)\, T(K_{n-k}).$$

Solution: It is possible to do this after substituting $T(K_n) = n^{n-2}$, but that will be a very difficult method.

We will count the following set in two ways.

$$H = \{(T, e) \mid e \text{ is an edge in a spanning tree } T \text{ of } K_n\}$$

Since there are $n-1$ edges in any tree, the cardinality of this set is the LHS. Note that deleting e from the tree T gives us two spanning trees T_1 and T_2 on a disjoint set of vertices.

Now choose a vertex $v \in K_n$. Now partition vertex set K_n as $[n] = A \sqcup ([n]\backslash A)$ such that $v \in A$. Make a spanning trees T_1 of vertices in A and another spanning tree T_2 of vertices in $[n]\backslash A$. Let $|A| = k$ and $|[n]\backslash A| = (n-k)$, then there are $k(n-k)$ ways to select a connecting edge e from T_1 to T_2. A can be chosen in $\binom{n-1}{k-1}$ ways. So we get that H equals $\bigsqcup_{v\in A\subseteq[n]} \{(T_1 + \{v_1, v_2\} + T_2, \{v_1, v_2\}) \mid T_1, T_2$ are spanning trees on $A, [n]\backslash A$ resp., $v_1 \in A, v_2 \in [n]\backslash A\}$.