Exercise Set #9

"Discrete Mathematics" (2025)

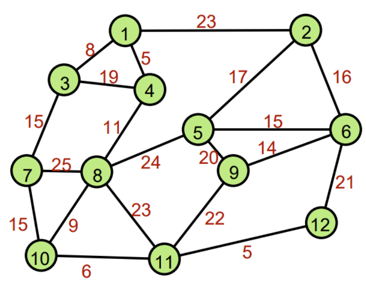
Exercise 7 is to be submitted on Moodle before 23:59 on April 28th, 2025

E1. Characterize the graphs (V, E) for which every connected subgraph (V', E') is equal to the induced subgraph on the respective set of vertices V'.

E2. Let G be a connected weighted graph. Assume it has at least one cycle, and let edge e be an edge that has strictly greater cost than all other edges in that cycle. Show that e does not belong to any minimal weight spanning tree of G.

E3. Prove that any connected graph with distinct weights assigned to the edges has a unique minimal weight spanning tree.

E4. Apply Kruskal's algorithm to the following graph to obtain a minimal spanning tree:



E5. Let $G = (V, E)$ be a weighted graph with the weight function $f : E \to \mathbb{R}$, i.e. the weight of an edge E is euqal to $f(E)$. What can you say about the output of Kruskal's algorithm for G (i.e. in terms of uniqueness or for part (3) and (4) how the output relates to the original output) if

(a) f is a constant function. That is for all $e \in E, f(e) = C$ for some fixed $C \in \mathbb{R}$.

(b) f is an injective function?

(c) f is replaced by the function $-f$ defined as $e \mapsto -f(e)$?

(d) f is replaced by a weight function $f' : E \to \mathbb{R}$ such that for all $e_1, e_2 \in E, f(e_1) \leq f(e_2) \Leftrightarrow f'(e_1) \leq f'(e_2)$?

E6. The following is called Prim's algorithm: Given a weighted graph G

- Initialize a tree with a single vertex, chosen arbitrarily from the graph.
- Grow the tree by one edge: of the edges that connect the tree to vertices not yet in the tree, find the minimum-weight edge, and add it to the tree.
- Repeat step 2 (until all vertices are in the tree).

Prove that the output of Prim's algorithm is a minimum spanning tree of G. Apply Prim's algorithm to the graph from Exercise 4.

E7. (**Exercise to submit**)

Let $G = (V, E)$ be a connected graph with edge weights $w : E \to \mathbb{Q}^+$, and suppose all edge weights are distinct. Let $\mathcal{T}$ be the set of all spanning trees of G, and let T^* be the *unique* MST. We say that a tree T is a second-best MST if

$$w(T) = \min_{A \in \mathcal{T} \setminus \{T^*\}} w(A)$$

(a) Let T^* be the MST of G. Prove that there exist edges $e, f \in E$ such that $T^* - e + f$ is a second-best MST.

(b) Show that the second-best MST is not necessarily unique.

(c) Consider T' a spawning tree that differs from the MST T^* by two or more edges. Show that T' cannot be a second-best MST.