Exercise Set #2

"Discrete Mathematics" (2025)

Exercise 8 is to be submitted on Moodle before 23:59 on March 3rd, 2025

E1. **(a)** Prove that $\binom{n}{k} \leq \binom{n}{k+1}$ when $1 \leq k < \lceil n/2 \rceil$.

(b) For which values of n and k is $\binom{n}{k+1}$ twice the previous entry in the Pascal Triangle (i.e. the entry to its left)?

E2. Prove the following equations.

(a)

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}.$$

(b)

$$\sum_{k=q}^{n} \binom{n}{k}\binom{k}{q} = 2^{n-q}\binom{n}{q}$$

E3. Show that $\left(4n^3 - 3n + 2023\right)\left(n^6 + 3n^4 + 256\right) = O\left(n^9\right)$

E4. Prove the following equations

(a) $n^a = O\left(a^n\right)$ for any $a > 1$.

(b) $n^a = O\left(n^b\right)$ for any $a \leq b$.

(c) $2^n + n^2 = O\left(2^n\right)$.

(d) $\frac{n}{\log n} = o(n)$.

E5. Suppose there are 20 students and each student has to choose a number in [100]. What are the chances that all students choose a different number? If the number of students is increased, does this probability increase or decrease? How many students should there be so that the probability that two students have chosen the same number is at least 50% ?

E6. Use Stirling's formula to estimate $1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)$.

E7. Find a combinatorial proof of the following identity

$$\sum_{\substack{k \leq n \\ k \text{ even}}} \binom{n}{k} = \sum_{\substack{k \leq n \\ k \text{ odd}}} \binom{n}{k}$$

E8. (**Exercise to submit**)
Find a combinatorial proof of the following identities

(a)

$$1+\binom{n}{1}2+\binom{n}{2}4+\cdots+\binom{n}{n-1}2^{n-1}+\binom{n}{n}2^n=3^n$$

(b)

$$\binom{n}{0}\binom{m}{k}+\binom{n}{1}\binom{m}{k-1}+\cdots+\binom{n}{k-1}\binom{m}{1}+\binom{n}{k}\binom{m}{0}=\binom{n+m}{k}.$$