Exercise Set #11

"Discrete Mathematics" (2025)

Exercise 7 is to be submitted on Moodle before 23:59 on May 12th, 2025

E1. In a kindergarden, there are 12 boys, 3 of whom are 3 years old, 5 are 4 years old and 4 are 5 years old; and 9 girls, 4 of whom are 3 years old, 2 are 4 years old and 3 are 5 years old. We pick one child, each with equal probability.

(1) What is the probability of picking a girl?

(2) What is the probability of picking a girl, provided that we pick a 3 years-old?

(3) What is the probability of picking a 3 years-old, provided that it is a girl?

Answer the same questions for boys.

E2. Let $\mathcal{F}$ be a family of 3 -element subsets of a finite set X. Prove that the elements of X can be colored with 3 colors so that at least $|\mathcal{F}|3!/3^3$ sets in $\mathcal{F}$ have exactly one element of each color.

E3. Compute the expected number of 3-cycles in a random graph on n vertices. Here the probability space is the set of all possible graphs (of which there are $2^{\binom{n}{2}}$ of those) and each random graph is assumed to be equally likely.

E4. Let $\{v_1, \ldots, v_n\}$ be unit vectors in $\mathbb{R}^d$. Prove that it is possible to choose signs $\varepsilon_i \in \{\pm 1\}$ such that the vector $\sum_{i=1}^n \varepsilon_i v_i$ has Euclidean norm less than or equal to $\sqrt{n}$.

E5. (1) For a graph $G = (V, E)$, we denote the complement of G as $G' = \left(V, \binom{V}{2} \backslash E\right)$. That is v_1, v_2 is an edge in G' if and only if $\{v_1, v_2\} \notin E$. We call a set $S \subset V(G)$ a clique if for two $s_1, s_2 \in S, \{s_1, s_2\} \in E(G)$. Then apply Turán's theorem on G' to prove the following equivalent version.
If G has n vertices but no cliques of size $r+1$, then

$$|E| \leq \frac{r-1}{r} \frac{n^2}{2}.$$

(2) For any given value $s, t \in \mathbb{Z}_{\geq 1}$, find a graph G_t on $n = s \cdot t$ vertices with $s \cdot t \cdot (t-1)/2$ edges such that $\alpha\left(G_t\right) = s$. Check that this is equal to the lower bound on independence number in Turán's theorem for each t.

E6. Find $\alpha(G)$ when G is one of the following graphs. Compare it with the lower bound given by Turán's theorem.

(1) The complete graph $K_n = \left([n], \binom{[n]}{2}\right)$. That is to say that K_n is a graph with n vertices such that there is an edge between any two vertices.

(2) The complete bipartite graph $K_{n,m} = ([n] \sqcup [m], [n] \times [m])$. That is, the vertices are into two groups of size n and m and there is an edge between each vertex of one group to another.

(3) A path graph $P_n = ([n], E)$ where $E = \{\{i, i+1\}\}_{i=1}^{n-1}$.

(4) A circular graph $C_n = P_n + \{1, n\}$.

E7. (**Exercise to submit**)
Let $A_1, \dots, A_n$ and $B_1, \dots, B_n$ be distinct subsets of $\mathbb{N}$ such that for each $i \in \{1, \dots, n\}$ we have $|A_i| = r$, $|B_i| = s$, and $A_i \cap B_i = \emptyset$; and for every $i \neq j$, $A_i \cap B_j \neq \emptyset$.

(a) Consider the universe $X = \bigcup_{i=1}^{n} A_i \cup B_i$. Take a random permutation of X and define the event X_i to be that all elements of A_i precede all elements of B_i in permutation. Show that $\mathbb{P}(X_i) = \binom{r+s}{r}^{-1}$

(b) Conclude that $n \leq \binom{r+s}{r}$.