Mock Exam 2025			
EPFL	**Course:**	Numerical Analysis – EL/MX/CH	
	Lecturer:	Prof. Michael Herbst	
	Date:	Spring Session 2025	**Duration:** 3h 00

Mock exam disclaimer

- This mock exam gives you an example how the final exam will look like. Questions 1–4, 7, & 8 have been taken from the **2024 exam**, question 5 & 6 from the **2024 mock exam**. As the course has changed compared to last year, some questions may use terminology we did not employ this year.
- In 2025, the final exam will only contain **pen and paper** questions, which moreover will cover the contents of **the entire class**. Furthermore you can expect this year's **final exam** to be a little **more involved** than this mock exam.

Exam instructions

This exam has **8 exercises** with in total **45 points**. The material consists of **two** parts:

- This **question sheet** with the questions. All questions are pen and paper questions.
- A personalised **answer sheet** (with your name and sciper number).

Answering the questions

- For each question on this question sheet, provide the **answers in the corresponding section** of the **answer sheet**. **Do not write onto the question sheet itself**. Only these answer sheets will be marked.
- On the **answer sheet** only **write within the black boxes**. If you need extra space, additional blank pages are given in the back. **Clearly identify for which question** you provide additional answers. **Also add a remark in the original answer box** where you run out of space that additional text can be found in the appendix.
- **Please write with a pen (no pencil or erasable ballpen)**.
- For each of your answers, **outline the reasoning** and **justify your answer**.
- At the end of the exam both the question and answer sheet will be collected.

Authorised material

- You are allowed **a two-sided handwritten A4 cheatsheet** (hand-written on paper, no print-outs).
- No other notes, sheets or books are allowed. No calculator, mobile phone, tablet, laptop or any other electronic device.

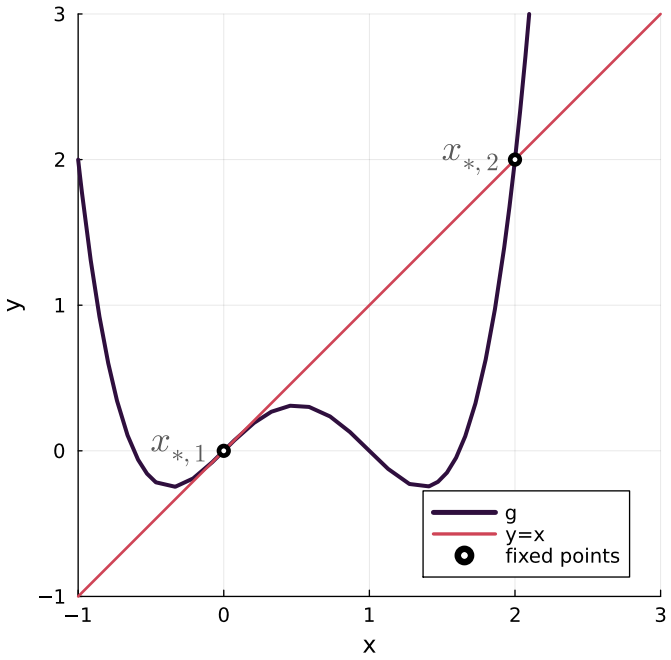
Exercise 1 (7P) — exam 2024

Given a parameter $\theta \in \mathbb{R}$, consider the function

$$g(x) = x^4 - 2\theta x^3 + x.$$

(a) **(2P)** Show that the only fixed points of g are $x_{*,1} = 0$ and $x_{*,2} = 2\theta$.

(b) **(1P)** Figure 1 depicts $g(x)$ for $\theta = 1$ together with its two fixed points $x_{*,1}$, $x_{*,2}$. By visual inspection determine for which of the two fixed points $x_{*,1}$ and $x_{*,2}$ the fixed-point iterations $x^{(k+1)} = g(x^{(k)})$ converge, provided that a starting point $x^{(0)}$ sufficiently close to the respective fixed point has been chosen. Justify your answer.



(c) **(3P)** We return to the general case where θ is a parameter of the problem.

(i) For which values of θ do fixed-point iterations converge to $x_{*,2}$ provided a good starting point is chosen?

(ii) For which values of θ is the fastest convergence rate to $x_{*,2}$ achieved ?

(d) We consider the case where θ is chosen such that the fastest convergence rate in the fixed-point iterations is achieved (the value you determined in (c) (ii)).

(1P) What is the convergence order of Newton's method for these value(s) of θ? Does Newton provide any advantage over fixed point iterations in this case?

Solution.

(a) We find the fixed points by solving the equation $g(x) = x$.

$$g(x) = x \iff x^4 - 2\theta x^3 + x = x$$
$$\iff x^4 - 2\theta x^3 = 0 \iff x^3(x - 2\theta) = 0$$

There are thus two solutions: $x_{*,1} = 0$ and $x_{*,2} = 2\theta$.

(b) The local convergence behaviour of the fixed point method is governed by the magnitude of the derivative at the fixed point. In order for the requested convergence to happen, one must have $|g'(x_{*,i})| < 1$ (*Theorem 1, Chapter 2*). On the given plot, this is clearly violated for $x_{*,2}$, while the situation seems to be borderline for $x_{*,1}$. We conclude that local convergence does not happen for $x_{*,2}$, while it should just happen for $x_{*,1}$. **(0.5 for correct answer, 0.5 for justification.)**

(c) According to the explanation above convergence is achieved if $|g'(x_{*,2})| < 1$ **(0.5P)**. Here, we have **(0.5P)**

$$g'(x) = 4x^3 - 6\theta x^2 + 1,$$

and so the condition for local convergence becomes **(0.5P)**:

$$|g'(x_{*,2})| < 1 \iff |8\theta^3 + 1| < 1,$$

which is satisfied for $\theta \in (-\frac{1}{\sqrt[3]{4}}, 0)$ **(0.5P)**.

According to (*Theorem 1, Chapter 2*), the convergence rate is given by $|g'(x_{*,i})|$ **(0.5P)**. In this case, the derivative at $x_{*,2}$ vanishes for $\theta = -1/2$, which thus provides the fastest convergence **(0.5P)**.

(d) According to (*Theorem 4, Chapter 2*), the convergence of Newton is only linear when $g'(x_*) = 0$. So that in the case of (c) the convergence order is identical to fixed point iterations.

Exercise 2 (8P) — exam 2024

(a) **(1.5P)** We are given $n+1$ nodes $x_1, \ldots, x_{n+1} \in \mathbb{R}$. Define the Lagrange basis associated to $x_1, \ldots, x_{n+1}$ and specify how this basis can be used to find the n-th degree interpolating polynomial through the data points $(x_1, y_1), (x_2, y_2), \ldots (x_{n+1}, y_{n+1})$.

(b) **(1.5P)** Using the Lagrange basis, find the interpolating polynomial through the points $(x_1 = -1, y_1 = -2)$, $(x_2 = 1, y_2 = 0)$, $(x_3 = 4, y_3 = 6)$.

For polynomial interpolation with equally spaced nodes, the interpolation error is governed by the following theorem:

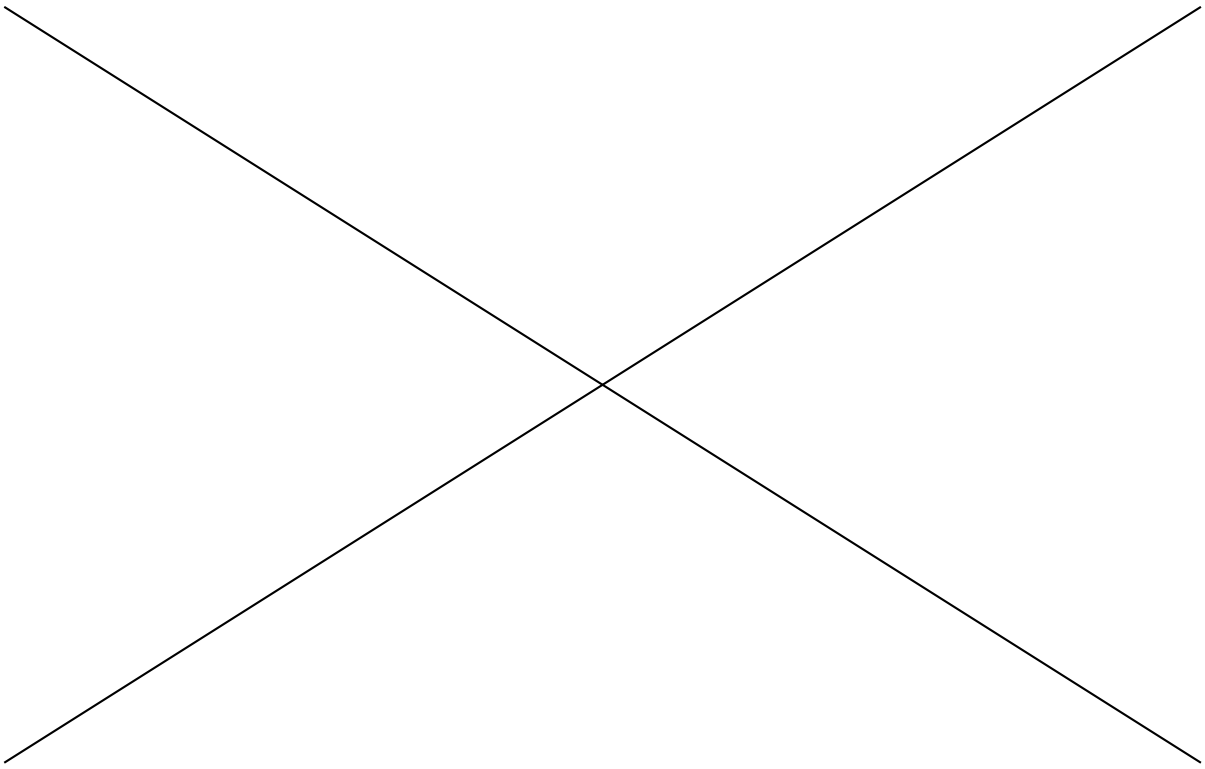
Theorem. For a C^{n+1} function $f : [a, b] \to \mathbb{R}$ and $a = x_1 < x_2 < \cdots < x_{n+1} = b$ equally distributed nodes in $[a, b]$ the n-th degree polynomial interpolant p_n of the data $(x_i, f(x_i))$ with $i = 1, 2, \ldots, n+1$ satisfies the following bound on the interpolation error

$$\max_{x \in [a,b]} |f(x) - p_n(x)| \leq \frac{1}{4(n+1)} \left(\frac{b-a}{n}\right)^{n+1} \max_{x \in [a,b]} \left| f^{(n+1)}(x) \right|. \tag{1}$$

We consider polynomial interpolation with $n+1$ equally distributed nodes over the interval $[-1, 1]$ for the functions $f_1(x) = \sin(x)$ and $f_2(x) = \frac{1}{1+20x^2}$.

(c) **(3P)** Show that for f_1 the interpolation error goes to 0 as $n \to \infty$.

(d) **(2P)** For f_2 we have $\max_{x \in [0,1]} \left| f_2^{(n+1)}(x) \right| \approx 20^n n!$ such that the right-hand side of (1) grows to infinity as $n \to \infty$. What happens to the polynomial interpolation in this case? What needs to be changed in the polynomial interpolation procedure to achieve exponential convergence for such functions f_2?



Solution.

(a) The Lagrange basis for the given set of nodes is (*Chapter 3, eq. (5)*) is made of the $n+1$ polynomials

$$L_i(x) = \prod_{\substack{j=1 \\ j\neq i}}^{n+1} \frac{x - x_j}{x_i - x_j}, \ i = 1, \dots, n+1.$$

Due to the cardinality condition **(0.5 P bonus)** this basis allows the n-th degree interpolating polynomial of the provided data to be directly written as:

$$p_n(x) = \sum_{i=1}^{n+1} y_i L_i(x).$$

(b) For the given points, we find:

$$\begin{aligned} p_2(x) &= -2\frac{x-1}{-2}\frac{x-4}{-5} + 0\frac{x+1}{2}\frac{x-4}{3} + 6\frac{x+1}{5}\frac{x-1}{3} \\ &= -\frac{1}{5}(x-1)(x-4) + \frac{2}{5}(x+1)(x-1) \\ &= \frac{1}{5}x^2 + x - \frac{6}{5}. \end{aligned}$$

(c) We begin by computing its $n+1$-th derivative **(1.5P)**:

$$f_1^{(n+1)}(x) = \begin{cases} (-1)^{n/2} \cos(x) & n+1 \text{ odd} \\ (-1)^{(n+1)/2} \sin(5x) & n+1 \text{ even.} \end{cases}$$

Therefore $\max_{x\in[a,b]} |f^{(n+1)}(x)| = 1$. Inserting this into the theorem we obtain **(1P)**

$$\max_{x\in[a,b]} |f_1(x) - p_n(x)| \leq \lim_{n\to\infty} \frac{1}{4(n+1)} \left(\frac{2}{n}\right)^{n+1} = \lim_{n\to\infty} \frac{2^{n+1}}{4n^{n+2}} = 0,$$

i.e. the interpolation error goes to zero as $n \to \infty$ **(0.5P)**.

(d) In the case of f_2 we run into Runge's phenomenon, i.e. that the interpolation error grows with larger polynomial degree n. A way to avoid this is to employ Chebyshev nodal points $x_k = -\cos\left(\frac{k\pi}{n}\right)$ for $k = 0, 1, \dots n$, which leads to exponential convergence (*Theorem 3, Chapter 3*).

Exercise 3 (7P) — exam 2024

We consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}$$

for $\alpha > 0$ and an associated linear system

$$\mathbf{A}\mathbf{x} = \mathbf{b} \quad \text{with} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2}$$

where we seek the solution $\mathbf{x} \in \mathbb{R}^2$.

When representing (2) on a computer we assume that the available floating-point precision is unable to represent $\mathbf{b}$ exactly introducing a small error $\varepsilon > 0$: the computer is only able to solve the approximate linear system

$$\mathbf{A}\widetilde{\mathbf{x}} = \widetilde{\mathbf{b}} \quad \text{with} \quad \widetilde{\mathbf{b}} = \begin{pmatrix} 1+\varepsilon \\ 1-\varepsilon \end{pmatrix} \tag{3}$$

and thus only able to obtain an approximate solution $\widetilde{\mathbf{x}} \in \mathbb{R}^2$.

(a) **(2P)** For a *general* square and invertible matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$ define the condition number $\kappa(\mathbf{M})$ in terms of matrix norms. Also provide an expression to compute the condition number using eigenvalues of appropriate matrices.

(b) **(2P)** Show that the condition number of $\mathbf{A}$ is

$$\kappa(\mathbf{A}) = \left|\frac{1+\alpha}{1-\alpha}\right|. \tag{4}$$

You may use that a matrix $\left(\begin{smallmatrix} a & c \\ c & b \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}$ has eigenvalues

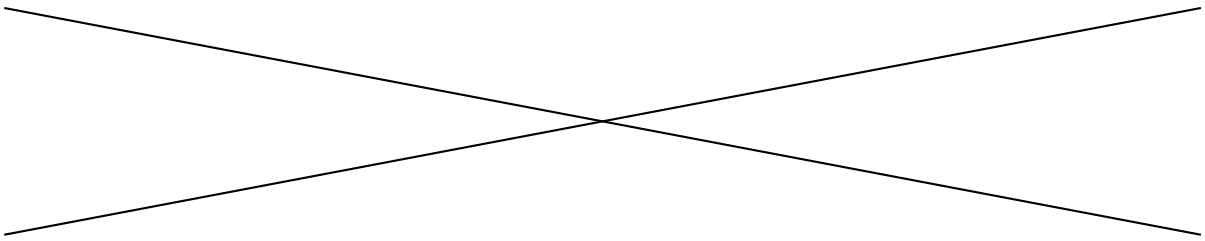
$$\frac{a+b}{2} - \sqrt{\frac{(a-b)^2}{4} + c^2} \qquad \text{and} \qquad \frac{a+b}{2} + \sqrt{\frac{(a-b)^2}{4} + c^2}.$$

(c) **(1P)** The solution to the perturbed system (3) is given by

$$\widetilde{\mathbf{x}} = \frac{1}{1+\alpha}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{\varepsilon}{1-\alpha}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and similarly $\mathbf{x} = \frac{1}{1+\alpha}\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$. Compute the relative error in the solution $\frac{\|\widetilde{\mathbf{x}}-\mathbf{x}\|}{\|\mathbf{x}\|}$ as well as the relative error in the right-hand side $\frac{\|\widetilde{\mathbf{b}}-\mathbf{b}\|}{\|\mathbf{b}\|}$.

(d) **(2P)** Describe in one sentence what the condition number measures for a linear system (2). Use this to explain your results in (c).



Solution.

(a) The condition is the quantity

$$\kappa(\mathbf{M}) = \left\|\mathbf{M}^{-1}\right\| \|\mathbf{M}\| = \frac{\sqrt{\lambda_{\max}(\mathbf{M}^T\mathbf{M})}}{\sqrt{\lambda_{\min}(\mathbf{M}^T\mathbf{M})}}$$

where $\lambda_{\max}(\mathbf{M}^T\mathbf{M})$ is the largest eigenvalue of $\mathbf{M}^T\mathbf{M}$ and $\lambda_{\min}(\mathbf{M}^T\mathbf{M})$ is the smallest eigenvalue of $\mathbf{M}^T\mathbf{M}$. *Award **(1P)** for each of the expressions.*

(b) We need to determine the eigenvalues of **(0.5P)**

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}\begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix} = \begin{pmatrix} 1+\alpha^2 & 2\alpha \\ 2\alpha & 1+\alpha^2 \end{pmatrix}.$$

Using the provided formula we determine the eigenvalues as **(0.5P)**

$$1 + \alpha^2 \pm 2\alpha$$

and thus the condition number as **(1P)**

$$\kappa(\mathbf{A}) = \sqrt{\frac{1+\alpha^2+2\alpha}{1+\alpha^2-2\alpha}} = \left|\frac{1+\alpha}{1-\alpha}\right|$$

$$\mathbf{x} = \frac{1}{1+\alpha}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(c) Since

$$\|\mathbf{x} - \widetilde{\mathbf{x}}\| = \left\|\frac{\varepsilon}{1-\alpha}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\| = \frac{\sqrt{2}\,\varepsilon}{|1-\alpha|} \qquad \text{and} \qquad \|\mathbf{x}\| = \left\|\frac{1}{1+\alpha}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\| = \frac{\sqrt{2}}{|1+\alpha|}$$

we obtain

$$\frac{\|\widetilde{\mathbf{x}} - \mathbf{x}\|}{\|\mathbf{x}\|} = \frac{|1+\alpha|}{|1-\alpha|}\varepsilon$$

Moreover we compute

$$\frac{\left\|\mathbf{b} - \widetilde{\mathbf{b}}\right\|}{\|\mathbf{b}\|} = \varepsilon.$$

In general:

- *0.5 points for the relative error of* $\mathbf{x}$*, but they do not get these if they forget the absolute values.*
- *0.5 points for that of* $\mathbf{b}$ *!*

(d) *Examples for the one-sentence answer:*

- The condition number measures how much the solution $\mathbf{x}$ is changed on a perturbation of the RHS $\mathbf{b}$. **(0.5P)**
- *A more precise answer:* The condition number relates the relative error in the right-hand side $\mathbf{b}$ to the relative error in the solution $\mathbf{x}$ **(1P)**.

In (c) we observe the condition number to appear in the expression of the relative error of the solution **(0.5P)**, i.e.

$$\frac{\|\widetilde{\mathbf{x}} - \mathbf{x}\|}{\|\mathbf{x}\|} = \kappa(\mathbf{A})\varepsilon = \kappa(\mathbf{A})\frac{\left\|\mathbf{b} - \widetilde{\mathbf{b}}\right\|}{\|\mathbf{b}\|}$$

thus agreeing perfectly with the expectation from theory **(0.5P)**. *(ii, 1 pt total) If* $\|\tilde{\mathbf{x}} - \mathbf{x}\|/\|\mathbf{x}\| = \kappa(\mathbf{A})\varepsilon$ *is written, either in (c) or (d), they get 0.5/1. If the relation with explicitly made with theory / point (i), they get the full point.*

Exercise 4 (5P) — exam 2024

Consider the algorithm for **LU factorization** given below.

Algorithm (LU factorisation).
Input: $\mathbf{A} \in \mathbb{R}^{n\times n}$,
Output: $\mathbf{U} \in \mathbb{R}^{n\times n}$, $\mathbf{L} \in \mathbb{R}^{n\times n}$

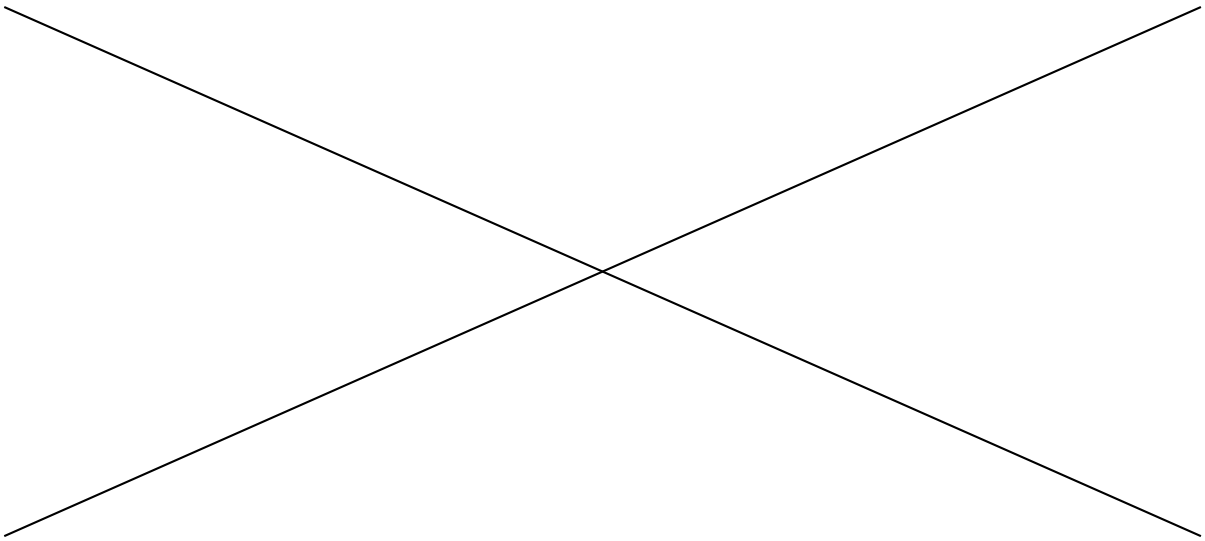
1. $\mathbf{A}^{(1)} = \mathbf{A}$
2. for $k = 1, \ldots, n-1$ *(algorithm steps)*
    1. $L_{kk} = 1$
    2. for $i = k+1, \ldots, n$ *(Loop over rows)*
        1. $L_{ik} = \frac{A_{ik}^{(k)}}{A_{kk}^{(k)}}$
        2. for $j = k+1, \ldots n$ *(Loop over columns)*
            1. $A_{ij}^{(k+1)} = A_{ij}^{(k)} - L_{ik} A_{kj}^{(k)}$
3. $\mathbf{U} = \mathbf{A}^{(n)}$

(a) **(1.5P)** Making reference to the algorithm explain why LU factorisation is said to have a computational cost of $\mathcal{O}(n^3)$.

(b) **(1.5P)** Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be an n by n square matrix. If $\mathbf{A}$ has no special structure what is the memory usage for storing $\mathbf{A}$? What is the memory usage for storing the $\mathbf{L}$ and $\mathbf{U}$ factors once LU factorisation has been performed? Specify your answer using big $\mathcal{O}$ notation.

(c) Now assume that $\mathbf{A}$ is sparse.

- **(1P)** If we only store the non-zero elements of the matrix explicitly, what is the memory cost of $\mathbf{A}$ in this case?
- **(1P)** Explain the phaenomenon called fill-in and its consequences for the memory cost of the $\mathbf{L}$ and $\mathbf{U}$ factors of sparse matrices.



Solution.

(a) We need to understand the cost of the innermost instruction and how many times it is executed. The innermost for LU factorisation is

$$A_{ij}^{(k+1)} = A_{ij}^{(k)} - L_{ik}A_{kj}^{(k)},$$

which contains a multiplication and a subtraction, two elementary operations **(0.5 P)**. This instruction is inside 3 nested loops (on k, i and j respectively) with the indexes going from 1 to n **(1 P)**. The total number of elementary operations is thus of the order of n^3. Then, the computational cost of the LU factorization of a full matrix is $\mathcal{O}(n^3)$.

(b) The storage cost of an $n \times n$ matrix, assuming no underlying structure, is $\mathcal{O}(n^2)$ **(0.5 P)**. In general, LU factorization does result in two matrices $\mathbf{L}$ and $\mathbf{U}$, which both store $\mathcal{O}(n^2)$ non-zero elements giving again rise to a cost of $\mathcal{O}(n^2)$ **(1 P)**. Overall the storage cost does not increase.

(c)
- We defined a sparse matrix as one that has only $\mathcal{O}(n)$ non-zero elements. Hence, if only those are stored, we end up with $\mathcal{O}(n)$ storage cost.
- If $\mathbf{A}$ is sparse, but has no other structure, then in general its LU factorization will not be sparse. So even for sparse matrices we will have a storage cost of $\mathcal{O}(n^2)$ for $\mathbf{L}$ and $\mathbf{U}$.

Exercise 5 (7P) — mock 2024

Let $\mathbf{Ax} = \mathbf{b}$ be a linear system with given $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{b} \in \mathbb{R}^n$. We consider the fixed-point map

$$g(\mathbf{x}) = \mathbf{x} + \mathbf{P}^{-1}\left(\mathbf{b} - \mathbf{Ax}\right) \tag{5}$$

for a invertible preconditioner matrix $\mathbf{P} \in \mathbb{R}^{n\times n}$. The iterative procedure $\mathbf{x}^{(k+1)} = g(\mathbf{x}^{(k)})$ starting from an initial vector $\mathbf{x}^{(0)} \in \mathbb{R}^n$ is the Richardson iteration.

(a) **(1P)** Show that if $\mathbf{x} \in \mathbb{R}^n$ is a fixed point of g than it is also a solution to the linear system $\mathbf{Ax} = \mathbf{b}$.

(b) **(2P)** Show that if $\mathbf{x}^{(k+1)} = g(\mathbf{x}^{(k)})$ and if $\mathbf{x}$ is a fixed point of g, then

$$(\mathbf{x}^{(k+1)} - \mathbf{x}) = (\mathbf{I} - \mathbf{P}^{-1}\mathbf{A})(\mathbf{x}^{(k)} - \mathbf{x}). \tag{6}$$

(c) **(1.5P)** Based on (6) give the conditions for Richardson iterations $\mathbf{x}^{(k+1)} = g(\mathbf{x}^{(k)})$ to converge to a fixed point $\mathbf{x}$ independent of the chosen initial vector $\mathbf{x}^{(0)}$ and right-hand side $\mathbf{b}$.

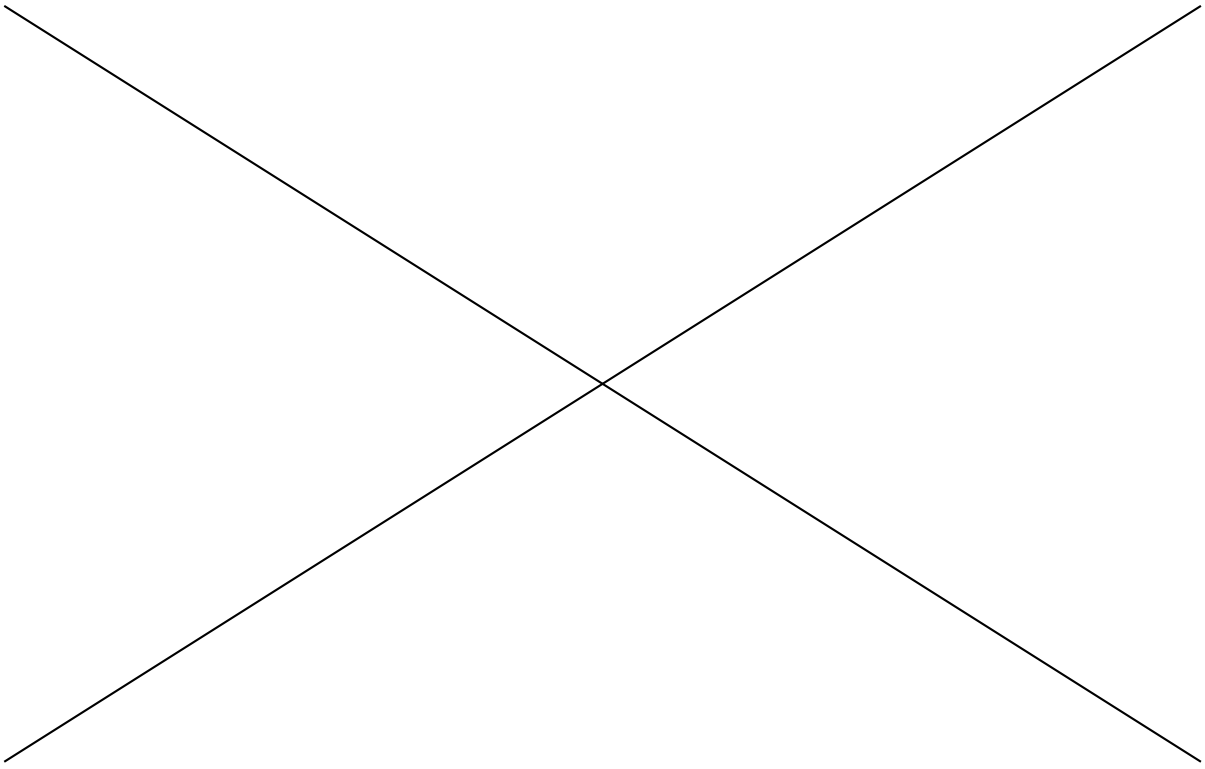
Consider the case

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & \alpha \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \mathbf{x}^{(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ and consider the fixed-point iterations $\mathbf{x}^{(k+1)} = g(\mathbf{x}^{(k)})$ for $k = 0, 1, \ldots$.

(d) **(1.5P)** For which choice of α and β do the fixed-point iterations converge.

(e) **(1P)** For which choice of α and β do the fixed-point iterations converge at the fastest possible rate. How may iteration steps are at most needed for convergence ?



Solution.

(a) If $\mathbf{x}$ is a fixed point of g, then $\mathbf{x} = g(\mathbf{x})$ **(0.5P)**, which implies

$$\mathbf{x} = \mathbf{x} + \mathbf{P}^{-1}(\mathbf{b} - \mathbf{A}\mathbf{x}) \quad \Leftrightarrow \quad \mathbf{0} = \mathbf{P}^{-1}(\mathbf{b} - \mathbf{A}\mathbf{x}) \quad \Leftrightarrow \quad \mathbf{b} = \mathbf{A}\mathbf{x}$$

as required **(0.5P)**.

(b) Since $\mathbf{x}^{(k+1)} = g(\mathbf{x}^{(k)})$ we have

$$\begin{aligned}\mathbf{x}^{(k+1)} - \mathbf{x} &= g(\mathbf{x}^{(k)}) - \mathbf{x} \\ &= \mathbf{x}^{(k)} + \mathbf{P}^{-1}(\mathbf{b} - \mathbf{A}\mathbf{x}^{(k)}) - \mathbf{x}\end{aligned}$$

Similarly expanding on the right we find

$$(\mathbf{I} - \mathbf{P}^{-1}\mathbf{A})(\mathbf{x}^{(k)} - \mathbf{x}) = \mathbf{x}^{(k)} - \mathbf{x} - \mathbf{P}^{-1}\mathbf{A}\mathbf{x}^{(k)} + \mathbf{P}^{-1}\mathbf{A}\mathbf{x}$$

Substracting both equations leads to

$$\mathbf{0} = \mathbf{P}^{-1}\mathbf{b} - \mathbf{P}^{-1}\mathbf{A}\mathbf{x}$$

which is true since $\mathbf{b} = \mathbf{A}\mathbf{x}$.

(c) The LHS of (6) is the error in the $k+1$-st iteration and the RHS is the error in the k-th iteration. Therefore if the matrix norm of $\mathbf{I} - \mathbf{P}^{-1}\mathbf{A}$ is less than 1, Richardson iterations converge independent of the starting vector $\mathbf{x}^{(0)}$.

(d) First we compute

$$\mathbf{I} - \mathbf{P}^{-1}\mathbf{A} = \begin{pmatrix} 1 - 2/\beta & 0 \\ 0 & 1 - \alpha \end{pmatrix}$$

This matrix is symmetric, hence it has matrix norm less than 1 iff its eigenvalues are less than 1 in magnitude. Since this is a diagonal matrix, its diagonal are the eigenvalues. We obtain the conditions

$$-1 < 1 - 2/\beta < 1 \qquad \text{and} \qquad -1 < 1 - \alpha < 1.$$

This is satisfied exactly if $0 < \beta < 1$ and $0 < \alpha < 2$.

(e) Fastest convergence is obtained for the smallest matrix norm. The choice $\alpha = 1$ an $\beta = 2$ leads to a matrix norm of zero **(0.5P)** (which is the smallest, since the matrix norm is non-negative), i.e. the iterations converge in a single step. **(0.5P)**

Exercise 6 (5P) — mock 2024

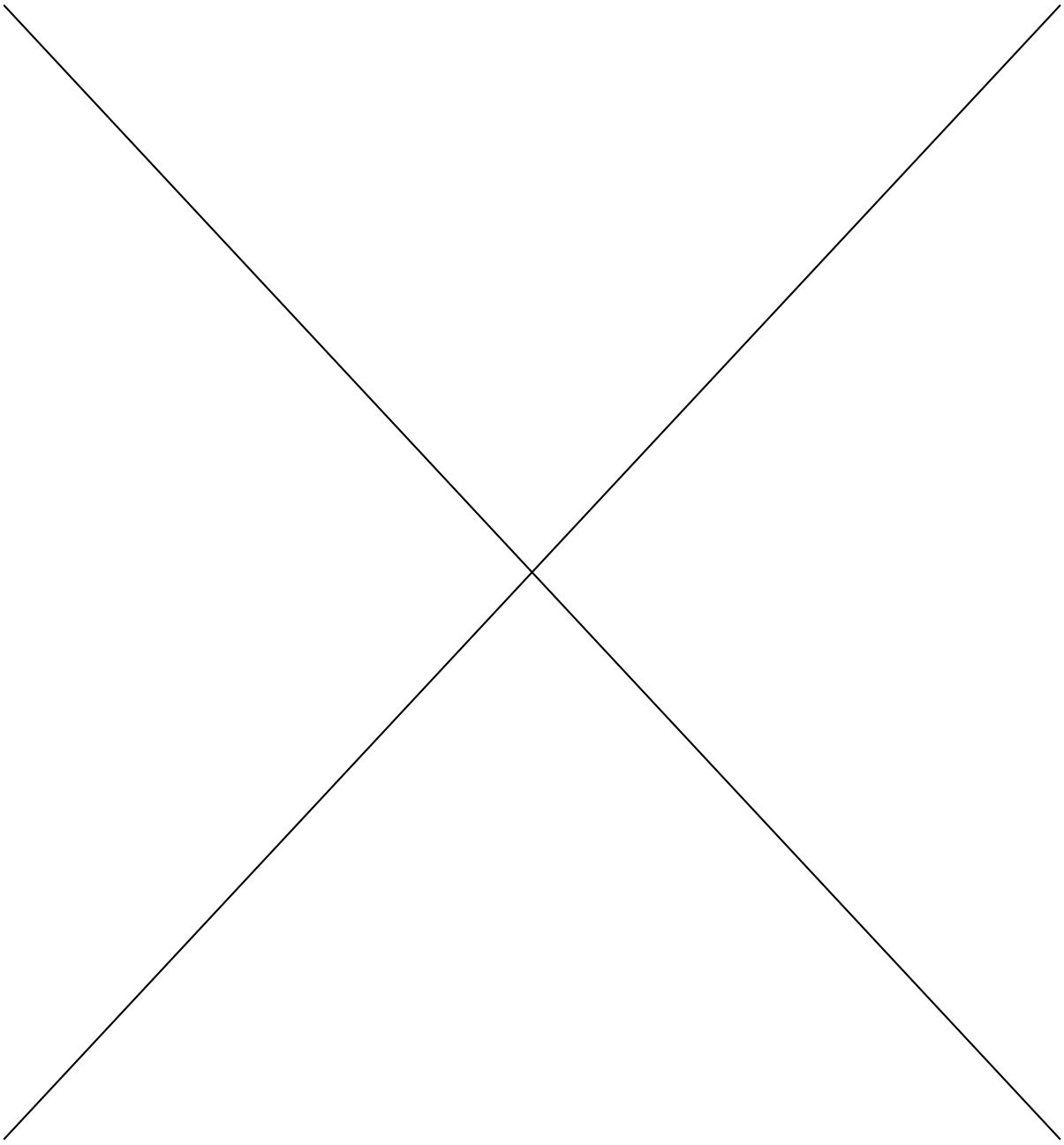
Consider $\mathbf{b} \in \mathbb{R}^2$ and the following 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -\beta \\ -\frac{1}{2} & 1 \end{pmatrix},$$

for a real number $\beta \in \mathbb{R}$. Consider Jacobi's method to solve the linear system $\mathbf{Ax} = \mathbf{b}$.

(a) **(2.5P)** State the conditions for Jacobi's method to converge for any right-hand side $\mathbf{b}$ and initial vector $\mathbf{x}^{(\mathbf{0})}$.

(b) **(2.5P)** Deduce conditions for the parameter β, which ensure convergence for any right-hand side $\mathbf{b}$ and initial vector $\mathbf{x}^{(\mathbf{0})}$.



Solution.

(a) Jacobi's method converges provided that the matrix norm of the iteration matrix is strictly smaller than 1 (this is a special case of Richardson's iterations). Mathematically, this means that Jacobi's method converges for the matrix $\mathbf{A}$, for any right-hand side $\mathbf{b}$ and any starting vector $\mathbf{x}^{(0)}$ if and only if:

$$\left\|\mathbf{I} - \mathbf{P}^{-1}\mathbf{A}\right\| < 1,$$

where $\mathbf{P}$ is the matrix containing the diagonal of $\mathbf{A}$ on the diagonal and 0 elsewhere (preconditioner for Jacobi's method).

(b) In our case, calling the iteration matrix $\mathbf{B}$,

$$\mathbf{B} = \mathbf{I} - \mathbf{P}^{-1}\mathbf{A} = \begin{pmatrix} 0 & \beta \\ \frac{1}{2} & 0 \end{pmatrix},$$

and since this matrix is not symmetric we need to use the general formula for the matrix norm

$$\|\mathbf{B}\| = \sqrt{\lambda_{\max}(\mathbf{B}^T\mathbf{B})}.$$

The matrix $\mathbf{B}^T\mathbf{B}$ is given by

$$\mathbf{B}^T\mathbf{B} = \begin{pmatrix} 0 & \frac{1}{2} \\ \beta & 0 \end{pmatrix}\begin{pmatrix} 0 & \beta \\ \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \beta^2 \end{pmatrix},$$

whose eigenvalues are $\frac{1}{4}$ and β^2.

Hence $\|\mathbf{B}\| = \max(\frac{1}{2}, |\beta|)$, and Jacobi's method converges for $\beta \in (-1, 1)$.

Exercise 7 (6P) — exam 2024

We consider the family of triangular matrices

$$\mathbf{A} = \begin{pmatrix} \lambda_1 & 1 & 1 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$.

(a) **(3P)** We consider the power iterations on the matrix $\mathbf{A}$ starting from an initial guess $\mathbf{x}^{(0)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. State conditions on the parameters λ_1, λ_2 and λ_3 for the power iterations to converge. To which eigenvalue will they converge? Specify the convergence order and provide the convergence rate in terms of λ_1, λ_2 and λ_3.

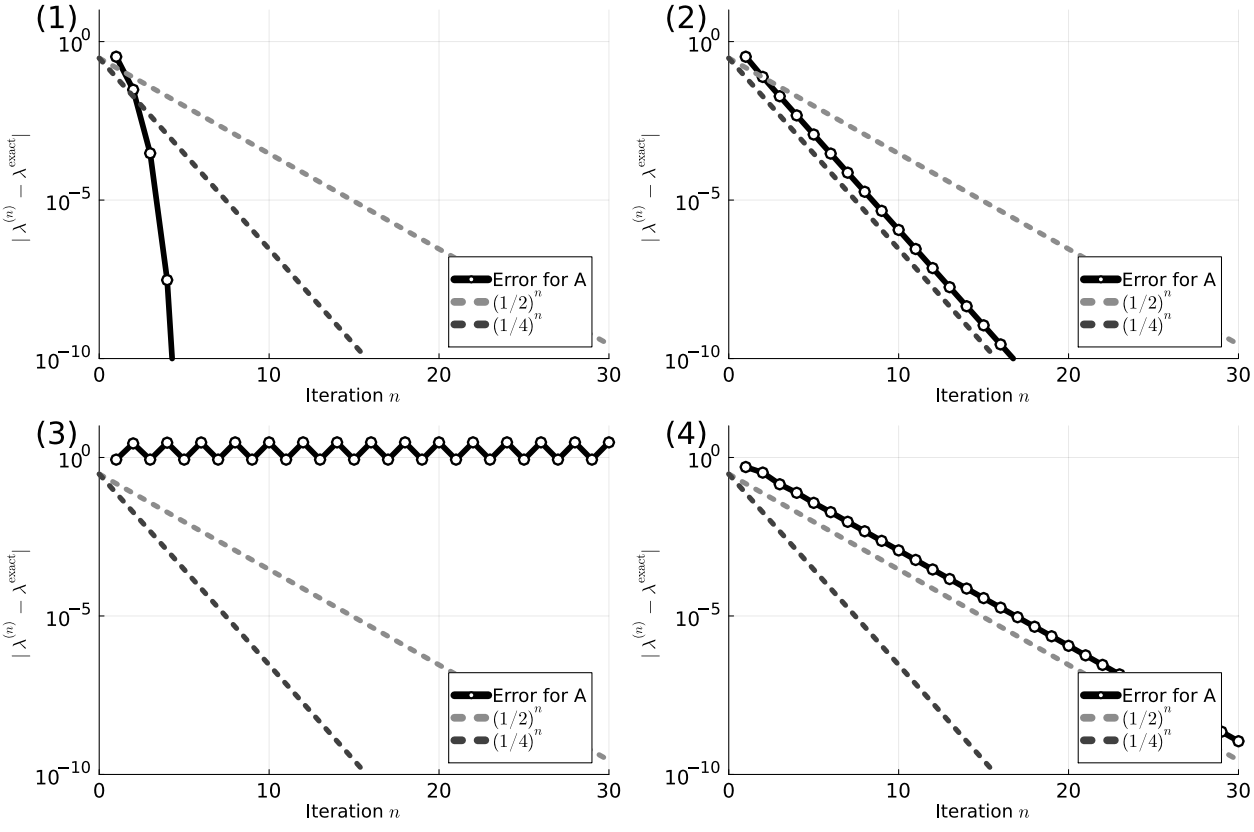
(b) **(1P)** Prove for a general matrix $\mathbf{M} \in \mathbb{R}^{n \times n}$:

> If $(\alpha, \mathbf{x})$ is an eigenpair of $\mathbf{M}$, i.e. $\mathbf{M}\mathbf{x} = \alpha\mathbf{x}$, and $\mathbf{M}$ is invertible, then $\left(\frac{1}{\alpha}, \mathbf{x}\right)$ is an eigenpair of $\mathbf{M}^{-1}$.

(c) **(2P)** Consider the matrix

$$\mathbf{B} = \begin{pmatrix} 5 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

which is a special case of $\mathbf{A}$ for $\lambda_1 = 5$, $\lambda_2 = 2$, $\lambda_3 = 1$. We perform *shifted inverse iterations* with shift $\sigma = 4$. Which eigenvalue λ_{exact} is targeted? Which of the following four plots is obtained? Justify your choice making reference to the discussion in the previous parts of the exercise.



Solution.

(a) The power iteration converges if the largest eigenvalue is a single eigenvalue **(0.5P)**, i.e. if $\lambda_1 > \lambda_2$ **(0.5P)**. All values of λ_3 can be employed since the condition $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$ ensures that λ_1 is dominating independent of λ_3 **(0.5P)**. The iterations will then converge to λ_1 (dominant eigenvalue) **(0.5P)**. Convergence is linear **(0.5P)** and the rate is $|\lambda_2/\lambda_1|$ **(0.5P)**.

(b) We multiply

$$\mathbf{M}\mathbf{x} = \alpha\mathbf{x}$$

on the left by $\mathbf{M}^{-1}$ to observe

$$\mathbf{x} = \alpha\mathbf{M}^{-1}\mathbf{x} \Leftrightarrow \frac{1}{\alpha}\mathbf{x} = \mathbf{M}^{-1}\mathbf{x}.$$

This shows that $(\frac{1}{\alpha}, \mathbf{x})$ is an eigenpair of $\mathbf{M}^{-1}$. *Note: If students mention that one can divide by α since $\mathbf{M}$ is invertible, hence $\alpha \neq 0$, then they get **(1P bonus)***

(c) When the shift is $\sigma = 4$, the eigenvalues of $(\mathbf{B} - \sigma)^{-1}$ are 1, $-\frac{1}{2}$ and $-\frac{1}{3}$. Hence the iterations converge to $\lambda_{\text{exact}} = 5$ **(0.5P)** with rate $1/2$ **(1P)**. The correct plot is thus (4) **(0.5P)**.

Exercise 8 (4P) — exam 2024

Let $f : [a, b] \to \mathbb{R}$ be a real-valued function with $0 < a < b$. We consider a numerical integration formula

$$Q(f) = h \sum_{i=0}^{n} w_i f(t_i)$$

with $n + 1$ equispaced quadrature nodes

$$t_i = a + ih \quad \text{for } i = 0, \ldots, n \qquad \text{and} \qquad h = \frac{b-a}{n}$$

as well as weights w_i for $i = 0, \ldots, n$. $Q(f)$ approximates $\int_a^b f(x)\,\mathrm{d}x$.

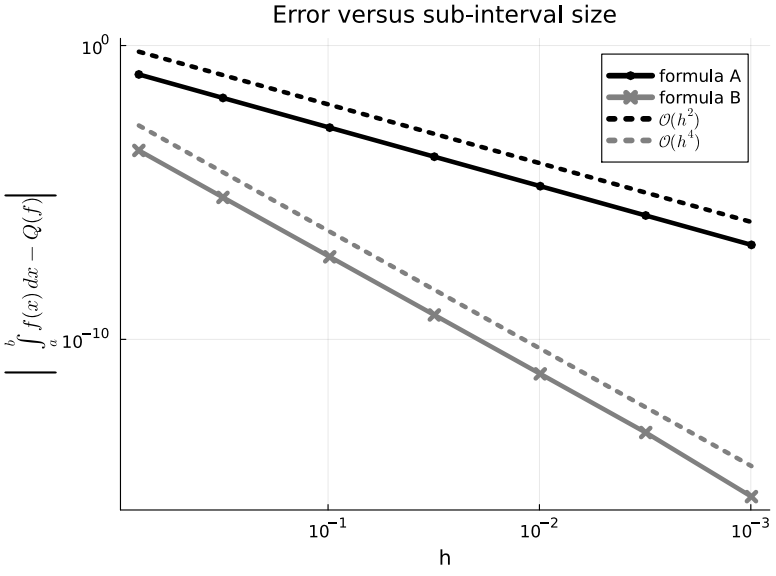
(a) **(0.5P)** Define the **degree of exactness** of Q.

(b) **(1P)** State the trapezoid formula for computing $\int_a^b f(x)\mathrm{d}x$ and provide its degree of exactness.

(c) **(2.5P)** In the lecture we discussed

> **Theorem.** If a numerical integration formula Q has a degree of exactness r then the formula is of order $r + 1$, i.e.
>
> $$\left| \int_a^b f(x)\,dx - Q(f) \right| \leq C\, h^{r+1}$$
>
> where C is a constant independent of h.

Inspect the following convergence graphs and apply this theorem to determine the degree of exactness of the two quadrature formulae (**Formula A** and **Formula B**). Which of the two formulae behaves like the trapezoid method?



Solution.

(a) The formula $Q(\,\cdot\,)$ has degree of exactness r if it integrates every polynomial with degree $\leq r$ exactly, but not a $(r+1)$-the degree polynomial. *If the "not a $(r+1)$-the degree polynomial" is missing award **(0P)**.*

(b) The trapezoid formula computes **(0.5P)**

$$T(f) = \frac{h}{2} f(a) + h \sum_{i=1}^{n-1} f(t_i) + \frac{h}{2} f(b).$$

It has degree of exactness 1 **(0.5P)**.

(c) **Formula A** converges with order 2, thus has degree of exactness 1 **(1P)** while **Formula B** converges with order 4, thus has degree of exactness 3 **(1P)**. The trapezoidal formula is thus **Formula A (0.5P)**.