Advanced Numerical Analysis

Lecture 5

Spring 2025

EPFL

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Quiz from Exercise Set 4

Given a function $f \in C^\infty([a,b])$, we consider the composite trapezoidal rule $Q_h^{(1)}[f]$ and the composite Simpson rule $Q_h^{(2)}[f]$ on the interval $[a,b]$. Which of the following statements are correct?

(i) $\lim_{h\to 0} Q_h^{(1)}[f] = \lim_{h\to 0} Q_h^{(2)}[f] = \int_a^b f(x)\,\mathrm{d}x$.

○ True ○ False

(ii) $|Q_h^{(1)}[f] - \int_a^b f(x)\,\mathrm{d}x| \leq |Q_H^{(1)}[f] - \int_a^b f(x)\,\mathrm{d}x|$ if $h \leq H$.

○ True ○ False

(iii) $|Q_h^{(2)}[f] - \int_a^b f(x)\,\mathrm{d}x| \leq |Q_h^{(1)}[f] - \int_a^b f(x)\,\mathrm{d}x|$ for suff. small $h > 0$.

○ True ○ False

(iv) If $f(x) \geq 0$ for all $x \in [a,b]$ then $0 \leq Q_h^{(1)}[f] \leq \int_a^b f(x)\,\mathrm{d}x$.

○ True ○ False

(v) If f is convex on $[a,b]$ then $Q_h^{(1)}[f] \geq \int_a^b f(x)\,\mathrm{d}x$.

○ True ○ False

Legendre polynomials

Theorem (Theorem 2.11)

The polynomial q_n defined by

$$q_n(x) = c_n \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n, \qquad c_n := \frac{1}{2^n n!},$$

is the nth Legendre polynomial.

Theorem (Theorem 2.12)

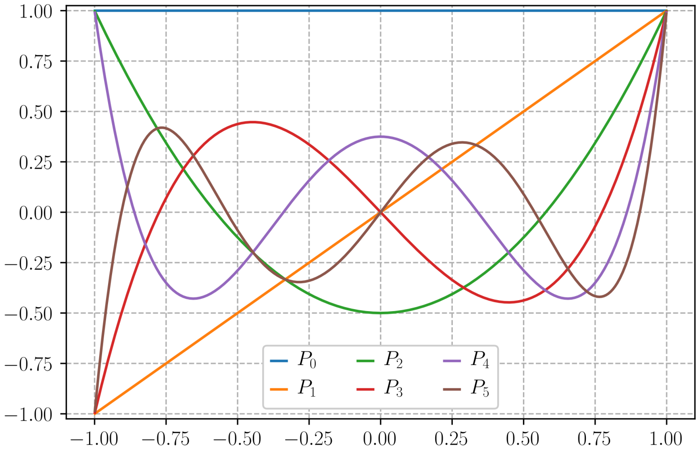
The Legendre polynomials $q_0, q_1, \ldots$ satisfy the three-term recurrence relation

$$q_{n+1}(x) = \frac{2n+1}{n+1} x q_n(x) - \frac{n}{n+1} q_{n-1}(x), \quad q_0(x) = 1, \quad q_1(x) = x.$$

Legendre polynomials

$$
\begin{aligned}
q_0(x) &= 1 \\
q_1(x) &= x \\
q_2(x) &= \frac{1}{2}(3x^2 - 1) \\
q_3(x) &= \frac{1}{2}(5x^3 - 3x) \\
q_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3) \\
q_5(x) &= \frac{1}{8}(63x^5 - 70x^3 + 15x).
\end{aligned}
$$

Legendre polynomials



The miracle of the trapezoidal rule

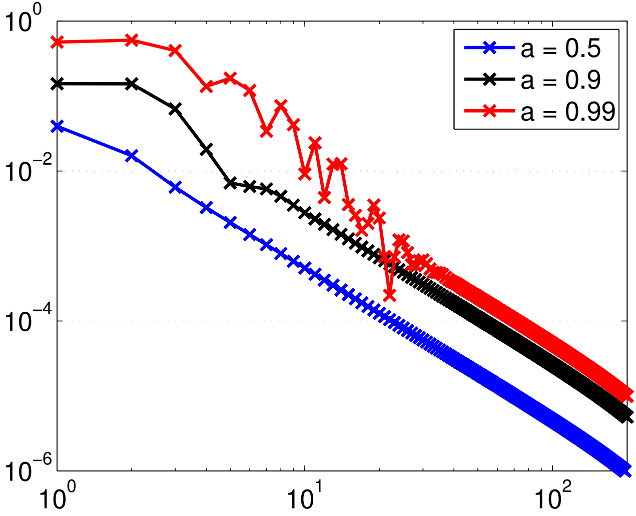
Consider the function

$$f(x) = \frac{1}{\sqrt{1 - a \cdot \sin(x-1)}}, \quad 0 < a < 1.$$

We have:

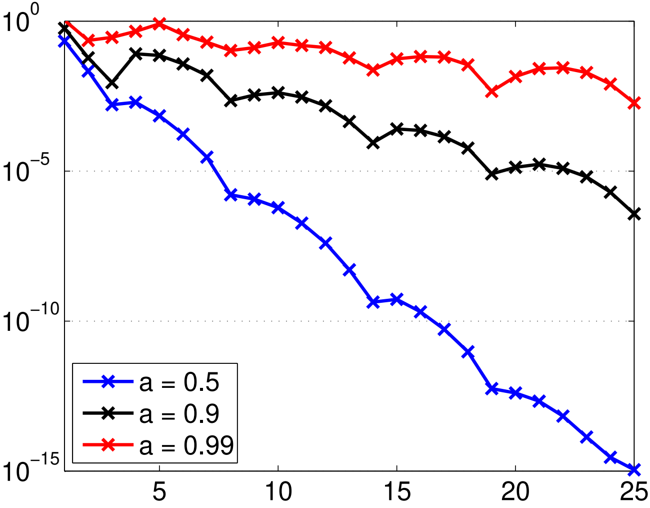
- f is periodic on the interval $[0, 2\pi]$.
- The periodic extension of f is smooth (infinitely often continuously differentiable), in fact, it is real analytic.
- As $a \to 1$, the function approaches a singularity at, e.g., $x = 2\pi/4 + 1$.

The miracle of the trapezoidal rule



Error vs. N for comp. trapezoidal applied to $\displaystyle\int_0^\pi f(x)\,\mathrm{d}x$

The miracle of the trapezoidal rule



Error vs. N for comp. trapezoidal applied to $\int_0^{2\pi} f(x)\, dx$