Advanced Numerical Analysis

Lecture 4

Spring 2025

EPFL

Daniel Kressner

Quiz from Exercise Set 3

a) Given interpolation points $x_0, x_1, \ldots, x_n \in [a, b]$, consider the operator $I : C^0([a, b]) \to \Pi_n$, $I : f \mapsto p_n$, which returns the polynomial p_n interpolating a given function f at the interpolation points. Which of the following statements are true about I?

(i) I is a linear operator

○ True ○ False

(ii) I is surjective

○ True ○ False

(iii) $|p_n(x)| \leq \max_{x\in[a,b]} |f(x)|$ for all $x \in [a, b]$

○ True ○ False

Recalling Lagrange polynomials

Given interpolation nodes $x_0, x_1, \ldots, x_n$, construct Lagange basis

$$\ell_j(x) := \prod_{i=0, i\neq j}^{n} \frac{x - x_i}{x_j - x_i}, \quad j = 0, \ldots, n.$$

Interpolating polynomial admits expression

$$p_n(x) = f_0\ell_0(x) + f_1\ell_1(x) + \cdots + f_n\ell_n(x).$$

Recalling Theorem 2.3

Theorem 2.3

Let $x_0 < x_1 < \cdots < x_n$ and let $p_n \in \mathbb{P}_n$ denote the interpolating polynomial satisfying $p_n(x_j) = f(x_j)$, $j = 0, \ldots, n$, for some function

$$f \in C^{n+1}([x_0, x_n]).$$

Given $x_\star \in [x_0, x_n]$, there exists $\xi \in [x_0, x_n]$ such that

$$E_n[f](x_\star) := f(x_\star) - p_n(x_\star) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,\omega_{n+1}(x_\star)\,, \tag{1}$$

where

$$\omega_{n+1}(x) := (x - x_0)(x - x_1)\cdots(x - x_n) \in \mathbb{P}_{n+1}\ . \tag{2}$$

Easier, relaxed version: Let $x_0, \ldots, x_n \in [a, b]$. Then

$$|f(x) - p_n(x)| \le \frac{|\omega_{n+1}(x)|}{(n+1)!} \max_{\xi \in [a,b]} |f^{(n+1)}(\xi)|, \quad \forall x \in [a, b].$$

Newton-Cotes formula

n		ξ_j	$\alpha_j/(b-a)$	Error
0	Midpoint rule	$\frac{1}{2}$	1	$\frac{1}{24}(b-a)^3 f^{(2)}(\xi)$
1	Trap. rule	0, 1	$\frac{1}{2}, \frac{1}{2}$	$-\frac{1}{12}(b-a)^3 f^{(2)}(\xi)$
2	Simpson rule	$0, \frac{1}{2}, 1$	$\frac{1}{6}, \frac{4}{6}, \frac{1}{6}$	$-\frac{1}{90}\left(\frac{b-a}{2}\right)^5 f^{(4)}(\xi)$
3	$\frac{3}{8}$ rule	$0, \frac{1}{3}, \frac{2}{3}, 1$	$\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}$	$-\frac{3}{80}\left(\frac{b-a}{3}\right)^5 f^{(4)}(\xi)$
4	Milne rule	$0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$	$\frac{7}{90}, \frac{32}{90}, \frac{12}{90}, \frac{32}{90}, \frac{7}{90}$	$-\frac{8}{945}\left(\frac{b-a}{4}\right)^7 f^{(6)}(\xi)$

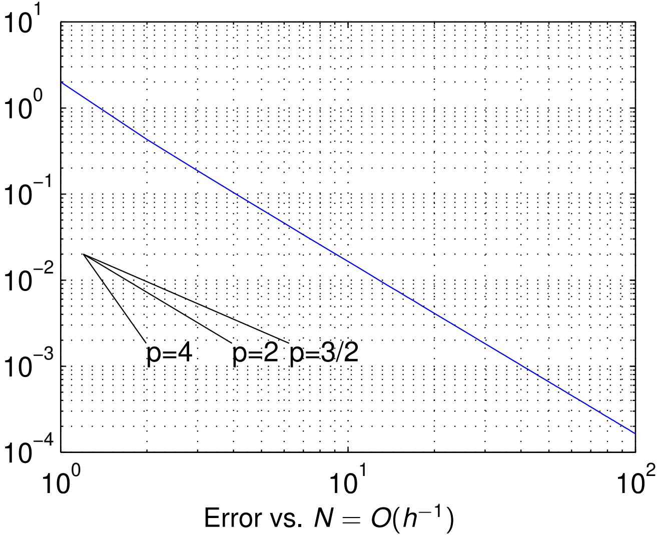
$$\int_a^b f(x)\,\mathrm{d}x \approx \sum_{j=0}^{n} \alpha_j f\left(a + \xi_j(b-a)\right)$$

Python implementation of composite trapezoidal rule

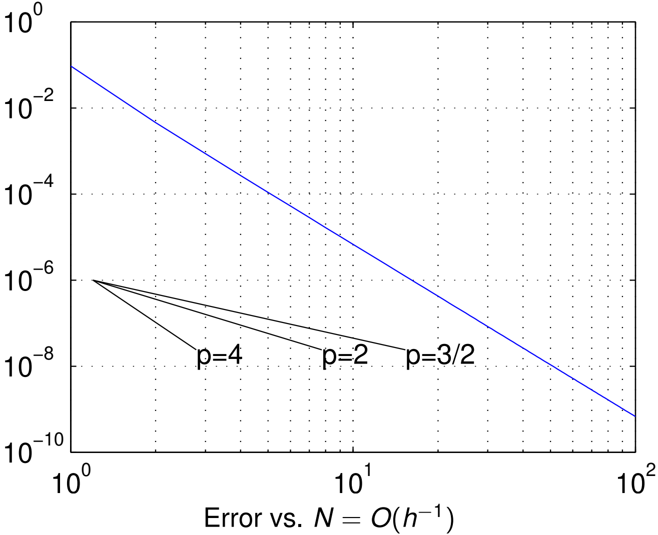
PYTHON

```
import numpy as np
import matplotlib.pyplot as plt

def trapez(fun, a, b, n):
  x = np.linspace(a,b,n+1)
  vecfun = np.vectorize(fun)
  f = vecfun(x)
  f[0] = f[0]/2
  f[-1] = f[-1]/2
  T = (b-a) * sum(f) / n
  return T

nn = 100; err = np.zeros(nn)
for n in range(nn):
  err[n] = np.abs(2 -
  trapez(np.sin, 0, np.pi, n + 1))
plt.loglog(range(1,nn+1), err)
```

Error of comp trapezoidal rule for $\int_0^\pi \sin(x)$



Error of comp Simpson rule for $\int_0^{\pi} \sin(x)$



Error of composite rules for $\int_0^1 \sqrt{x}$