Advanced Numerical Analysis

Lecture 3

Spring 2025

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Quiz from Exercise Set 2

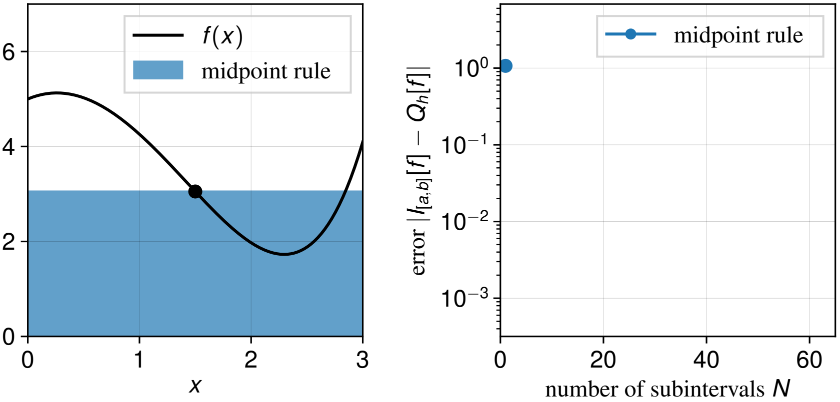
a) Consider the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$, which is known to diverge. When attempting to compute the partial sum $1 + 1/2 + 1/3 + \ldots + 1/n$ (from the smallest to the largest) in double precision, what will happen as $n \to \infty$?

- ◯ The computed partial sums will overflow.
- ◯ The computed partial sums will stagnate (“converge”) to ≈ 34.
- ◯ The computed partial sum will stagnate (“converge”) to $\approx 2 \times 10^{16}$.
- ◯ The computed partial sum will stagnate (“converge”) to $\approx 10^{300}$.

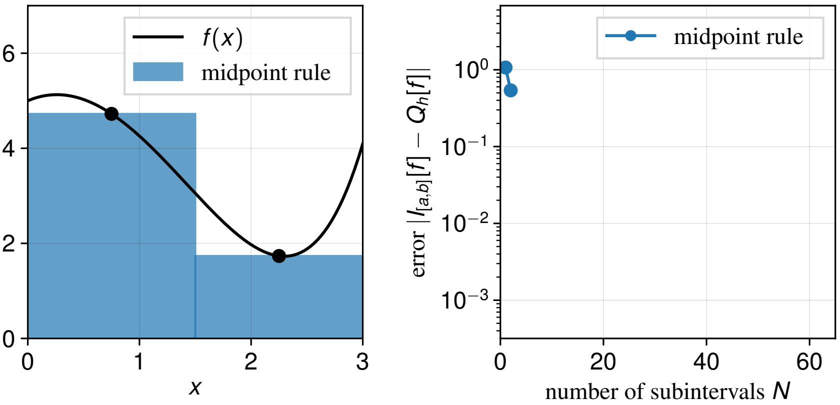
b) Consider the same question for the alternating harmonic series $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k}$, which is known to converge to $\log(2)$.

- ◯ The computed partial sums will overflow.
- ◯ The computed partial sums will stagnate (“converge”) to $\approx \log(2)$.
- ◯ The computed partial sums will underflow.
- ◯ The computed partial sum will stagnate (“converge”) to ≈ 0.

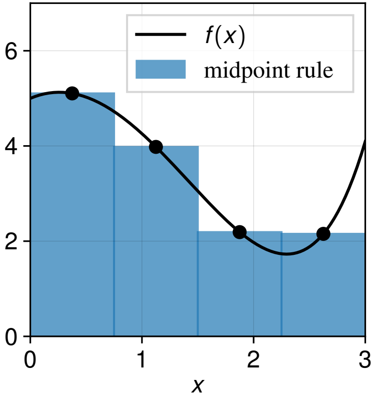
Midpoint rule

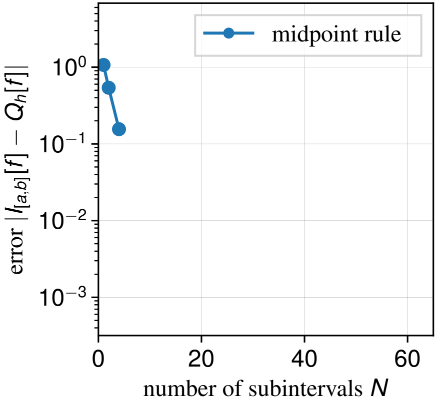


Midpoint rule



Midpoint rule





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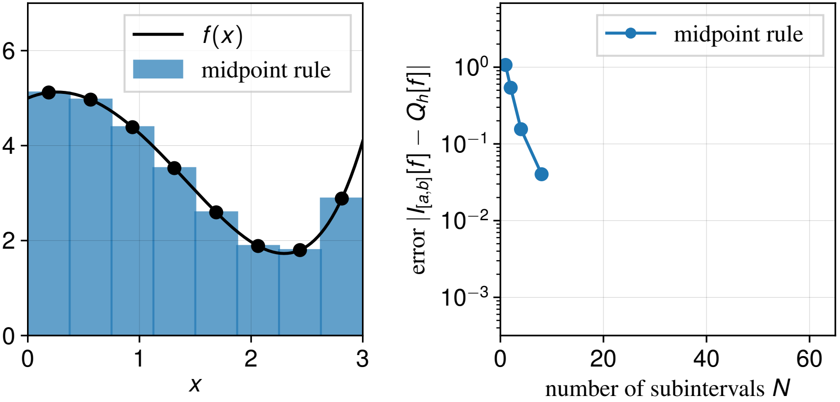

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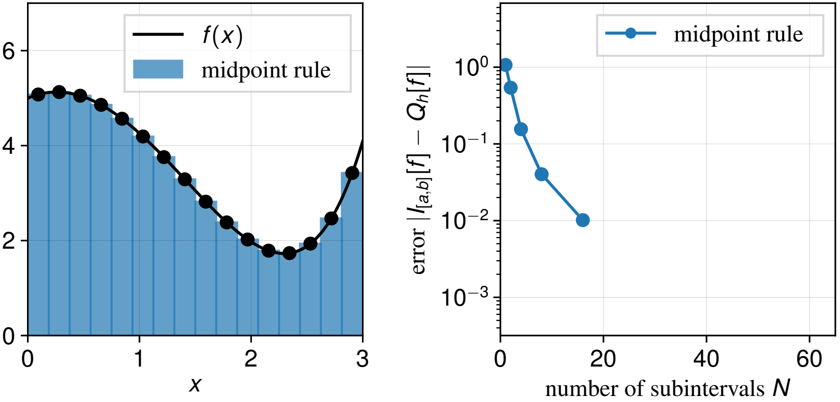

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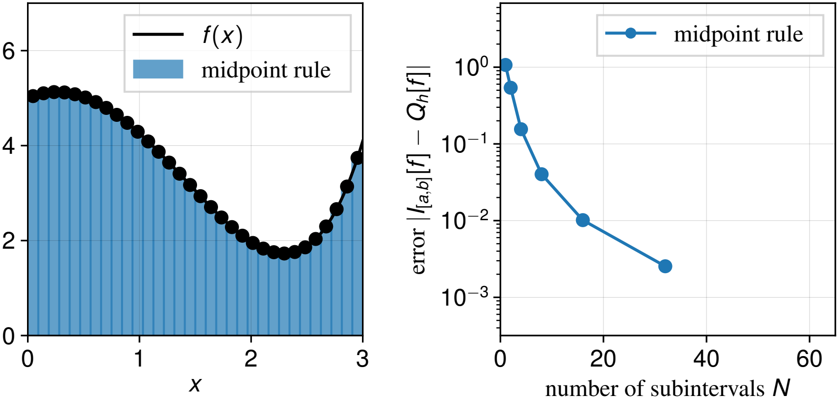

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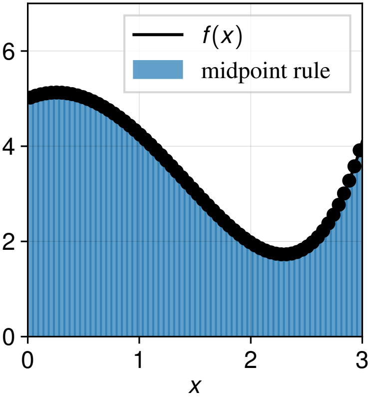
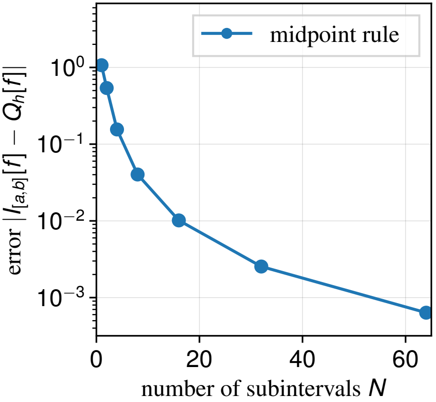
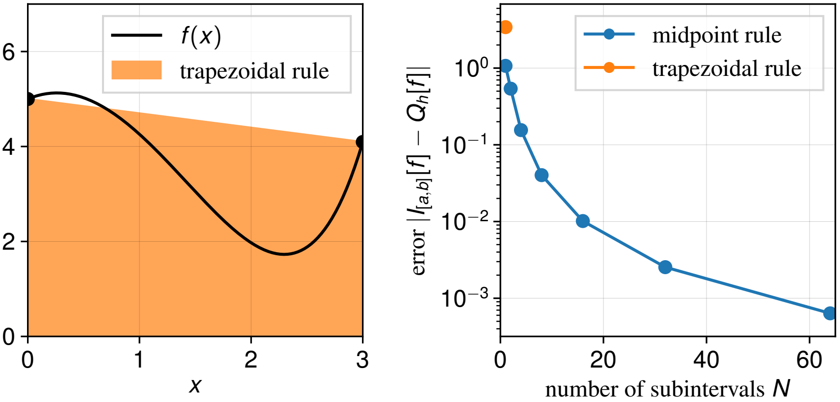
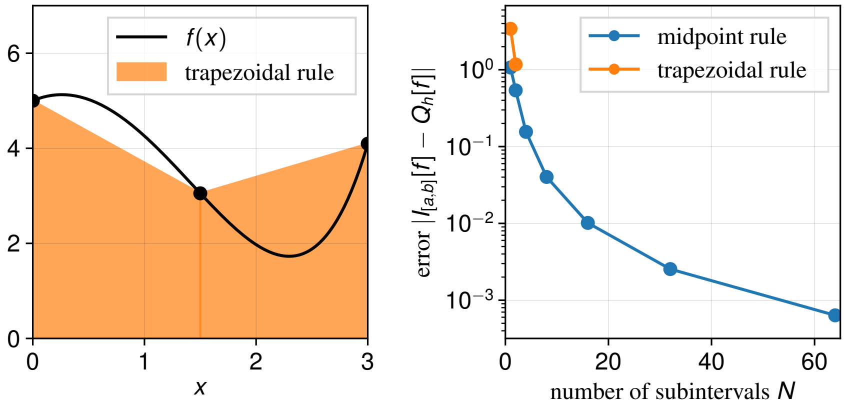




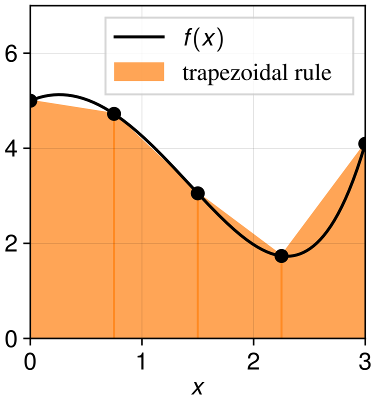
Trapezoidal rule

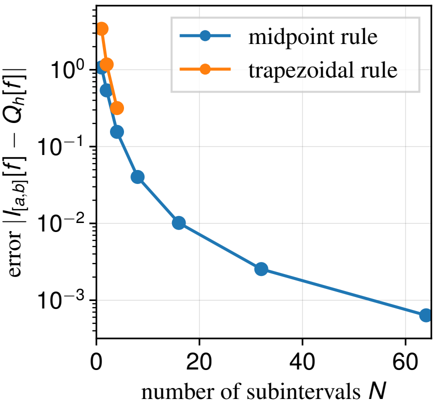


Trapezoidal rule

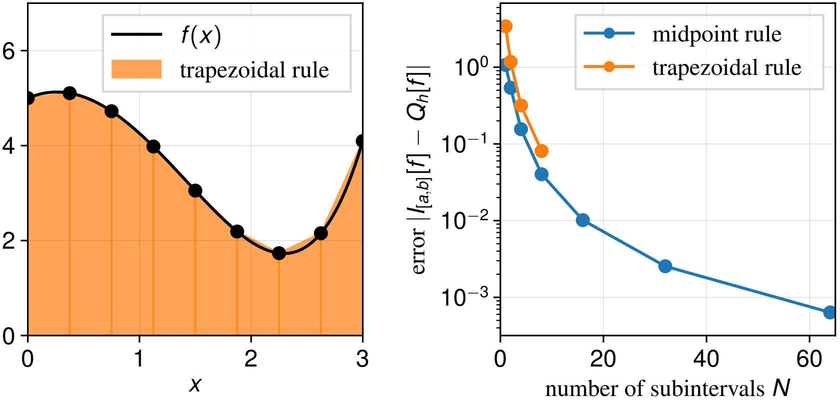


Trapezoidal rule

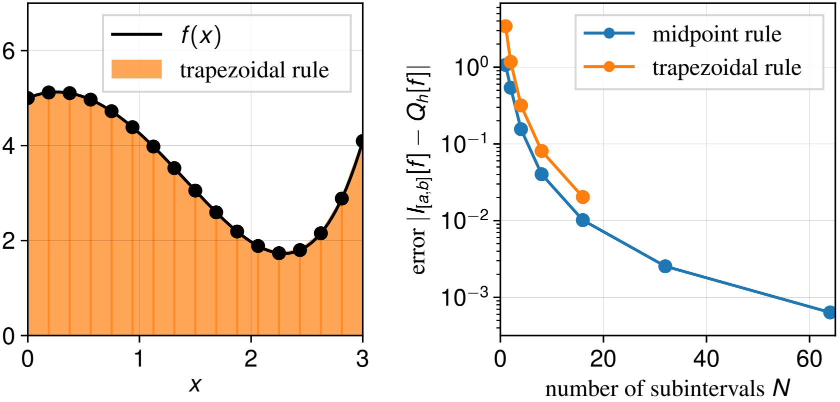




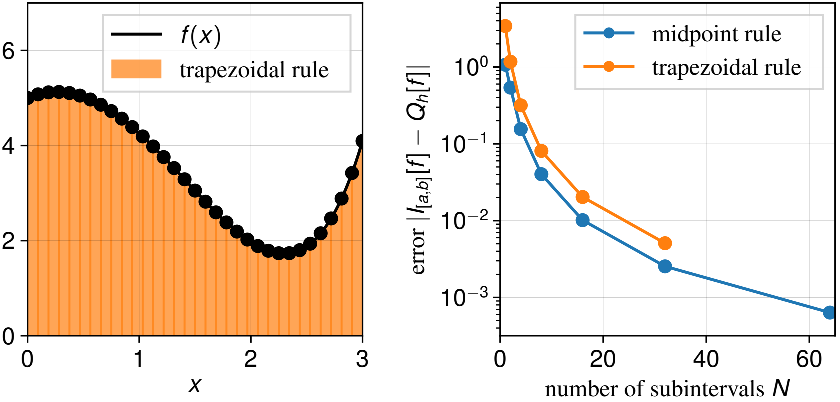
Trapezoidal rule



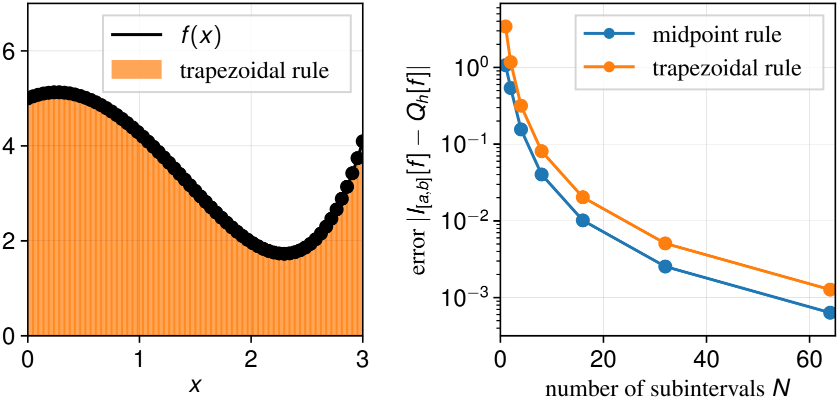
Trapezoidal rule



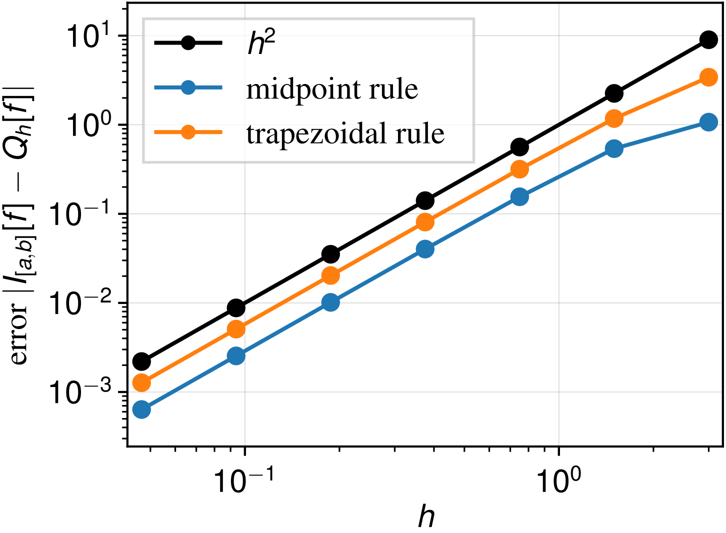
Trapezoidal rule



Trapezoidal rule



Convergence



Vandermonde matrix

Coefficients can be obtained by solving linear system involving the Vandermonde matrix

$$V = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}$$

- Is numerically problematic
- Still the standard way in Python (NumPy)

```
def polyfit(x, y, deg, rcond=None, full=False, w=None, cov=False):
    # set up least squares equation for powers of x
    lhs = vander(x, order)
    rhs = y

    # apply weighting
    if w is not None:
        w = NX.asarray(w) + 0.0
        if w.ndim != 1:
            raise TypeError("expected a 1-d array for weights")
        if w.shape[0] != y.shape[0]:
            raise TypeError("expected w and y to have the same length")
        lhs *= w[:, NX.newaxis]
        if rhs.ndim == 2:
            rhs *= w[:, NX.newaxis]
        else:
            rhs *= w

    # scale lhs to improve condition number and solve
    scale = NX.sqrt((lhs*lhs).sum(axis=0))
    lhs /= scale
    c, resids, rank, s = lstsq(lhs, rhs, rcond)
    c = (c.T/scale).T  # broadcast scale coefficients
```

$\longrightarrow$ Vandermonde matrix (for `lhs = vander(x, order)`)

$\longrightarrow V^{-1}x$ (for `lstsq(lhs, rhs, rcond)`)

Lagrange basis polynomials