Advanced Numerical Analysis

Lecture 12

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Audio signals

Let **y** contain an audio signal.

DFT $\mathbf{z} = F_n \mathbf{y}$ is called the (frequency) spectrum.

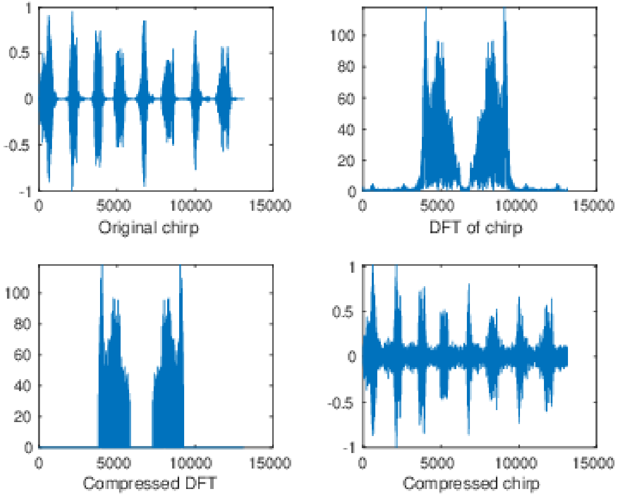
Important audio filtering operations proceed by manipulating **z**, followed by a back transformation to the time domain using inverse DFT.

Examples:

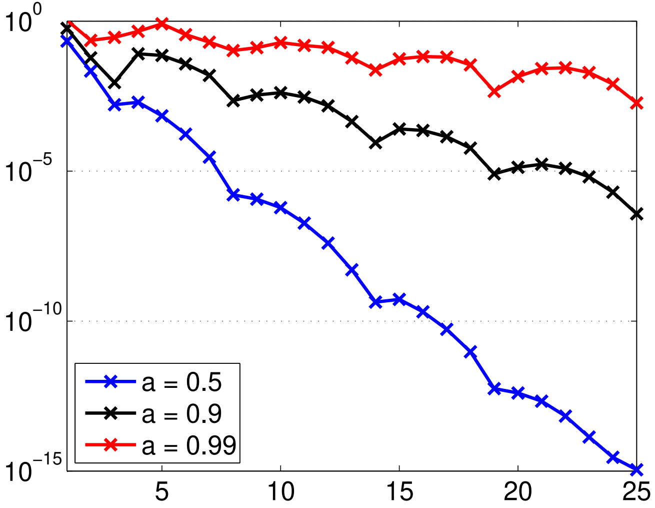
- Low-pass filter: Set high frequencies to zero ($z_j \leftarrow 0$ for larger j), preserve low frequencies.
- High-pass filter: Set low frequencies to zero ($z_j \leftarrow 0$ for smaller j), preserve high frequencies.
- Compression: Set $z_j \leftarrow 0$ if $|z_j| \leq \text{tol}$.

mp3

Audio signals



The trapezoidal miracle



Error vs. N for comp. trapezoidal rule $\int_0^{2\pi} f(x)\,dx$ for smooth periodic f

The trapezoidal miracle resolved

For 2π-periodic f, consider approximation of $\int_0^{2\pi} f(x)\,\mathrm{d}x$. Because of periodicity, composite trapezoidal rule takes the form

$$Q_h^{(1)}[f] = \frac{2\pi}{N} \sum_{j=0}^{N-1} f(j/N)$$

with $h = 2\pi/N$. Let us now consider truncated Fourier expansion

$$f_{N-1}(x) := \sum_{k=-N+1}^{N-1} c_k e^{\mathrm{i}kx}.$$

On the one hand, we have

$$\int_0^{2\pi} e^{\mathrm{i}kx}\,\mathrm{d}x = \begin{cases} 0 & \text{for } k \neq 0, \\ 2\pi & \text{for } k = 0. \end{cases}$$

On the other hand, Lemma 7.5 shows

$$Q_h^{(1)}[e^{\mathrm{i}k\cdot}] = \frac{2\pi}{N} \sum_{j=0}^{N-1} e^{2\pi \mathrm{i} jk/N} = \begin{cases} 0 & \text{for } k = -N+1, \ldots, -1, 1, \ldots, N-1, \\ 2\pi & \text{for } k = 0. \end{cases}$$

The trapezoidal miracle resolved

Linearity $\rightsquigarrow$ composite trapezoidal rule with $h = 2\pi/N$ integrates truncated Fourier expansion f_N *exactly*!

Error bound:

$$\begin{aligned} \left| \int_0^{2\pi} f(x)\,\mathrm{d}x - Q_h^{(1)}[f] \right| &\leq \left| \int_0^{2\pi} (f(x) - f_N(x))\,\mathrm{d}x - Q_h^{(1)}[f - f_N] \right| \\ &\leq 4\pi \sum_{|k|>N} |c_k|. \end{aligned}$$

For real analytic 2π-periodic function, Analysis IV told you that $|c_k|$ decays exponentially fast and, in turn, the error of the composite trapezoidal rule also converges exponentially fast to zero!

FFT (Fast Fourier Transform)

Some random facts:

- Published in 1965 by James Cooley and John Tukey.[1]
- One of MATH@EPFL seminar rooms named after John Tukey.
- In 1994, Gilbert Strang described the FFT as “the most important numerical algorithm of our lifetime”
- Our former president worked on it!

FAST FOURIER TRANSFORMS: A TUTORIAL REVIEW AND A STATE OF THE ART

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[1] But, of course, Gauss did it before, in 1805.

FFT (Fast Fourier Transform)

Let n be even (later: power of 2). Then $\omega_n = e^{-2\pi \mathrm{i}/n}$ satisfies

$$\omega_n^k = \omega_n^{k+n} \; \forall k \in \mathbb{Z}, \quad \omega_n^n = 1, \quad \omega_n^{n/2} = -1.$$

Most important relation is trivial: $\omega_n^{2k} = \omega_{n/2}^k$ This induces a LOT of structure in the DFT matrix

$$F_n = \left[\omega_n^{jk}\right]_{j,k=0}^{n-1}.$$

$$F_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega_6 & \omega_6^2 & \omega_6^3 & \omega_6^4 & \omega_6^5 \\ 1 & \omega_6^2 & \omega_6^4 & \omega_6^6 & \omega_6^8 & \omega_6^{10} \\ 1 & \omega_6^3 & \omega_6^6 & \omega_6^9 & \omega_6^{12} & \omega_6^{15} \\ 1 & \omega_6^4 & \omega_6^8 & \omega_6^{12} & \omega_6^{16} & \omega_6^{20} \\ 1 & \omega_6^5 & \omega_6^{10} & \omega_6^{15} & \omega_6^{20} & \omega_6^{25} \end{pmatrix},$$

Even rows $(0, 2, 4)$ and feature even powers!

FFT (Fast Fourier Transform)

Idea: Put even rows first with perfect shuffle permutation:

$$P_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$P_6^T \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_0 \\ x_2 \\ x_4 \\ x_1 \\ x_3 \\ x_5 \end{pmatrix}$$

FFT (Fast Fourier Transform)

$$P_6^T F_6 = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega_6^2 & \omega_6^4 & \omega_6^6 & \omega_6^8 & \omega_6^{10} \\ 1 & \omega_6^4 & \omega_6^8 & \omega_6^{12} & \omega_6^{16} & \omega_6^{20} \\ \hline 1 & \omega_6 & \omega_6^2 & \omega_6^3 & \omega_6^4 & \omega_6^5 \\ 1 & \omega_6^3 & \omega_6^6 & \omega_6^9 & \omega_6^{12} & \omega_6^{15} \\ 1 & \omega_6^5 & \omega_6^{10} & \omega_6^{15} & \omega_6^{20} & \omega_6^{25} \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega_6^2 & \omega_6^4 & 1 & \omega_6^2 & \omega_6^4 \\ 1 & \omega_6^4 & \omega_6^8 & 1 & \omega_6^4 & \omega_6^8 \\ \hline 1 & \omega_6 & \omega_6^2 & -1 & -\omega_6 & -\omega_6^2 \\ 1 & \omega_6^3 & \omega_6^6 & -1 & -\omega_6^3 & -\omega_6^6 \\ 1 & \omega_6^5 & \omega_6^{10} & -1 & -\omega_6^5 & -\omega_6^{10} \end{array}\right),$$

FFT (Fast Fourier Transform)

Because of $\omega_3^k = \omega_6^{2k}$ for $k \in \mathbb{Z}$, it follows that

$$P_6^T F_6 = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega_3^1 & \omega_3^2 & 1 & \omega_3^1 & \omega_3^2 \\ 1 & \omega_3^2 & \omega_3^4 & 1 & \omega_3^2 & \omega_3^4 \\ \hline 1 & \omega_6 & \omega_6^2 & -1 & -\omega_6 & -\omega_6^2 \\ 1 & \omega_6\omega_3^1 & \omega_6^2\omega_3^2 & -1 & -\omega_6\omega_3^1 & -\omega_6^2\omega_3^2 \\ 1 & \omega_6\omega_3^2 & \omega_6^2\omega_3^4 & -1 & -\omega_6\omega_3^2 & -\omega_6^2\omega_3^4 \end{array}\right) = \begin{pmatrix} F_3 & F_3 \\ F_3\Omega_3 & -F_3\Omega_3 \end{pmatrix}$$

with

$$F_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega_3^1 & \omega_3^2 \\ 1 & \omega_3^2 & \omega_3^4 \end{pmatrix}, \quad \Omega_3 = \begin{pmatrix} 1 & & \\ & \omega_6 & \\ & & \omega_6^2 \end{pmatrix}.$$

Theorem 10.7

Let $n \geq 2$ be even. Let P_n be the permutation matrix belonging to the permutation $\xi : \{0, \ldots, n-1\} \to \{0, \ldots, n-1\}$ with

$$\xi : 0 \mapsto 0,\ 1 \mapsto 2,\ \ldots,\ \frac{n}{2}-1 \mapsto n-2,\ \frac{n}{2} \mapsto 1,\ \frac{n}{2}+1 \mapsto 3,\ \ldots,\ n-1 \mapsto n-1.$$

Then

$$P_n F_n = \begin{pmatrix} F_{n/2} & F_{n/2} \\ F_{n/2}\Omega_{n/2} & -F_{n/2}\Omega_{n/2} \end{pmatrix} = \begin{pmatrix} F_{n/2} & \\ & F_{n/2} \end{pmatrix} \begin{pmatrix} I_{n/2} & I_{n/2} \\ \Omega_{n/2} & -\Omega_{n/2} \end{pmatrix}$$

with

$$\Omega_{n/2} = \mathrm{diag}(\omega_n^0, \omega_n^1, \ldots, \omega_n^{n/2-1}).$$