Advanced Numerical Analysis

Lecture 11
Spring 2025

EPFL

Daniel Kressner

Quiz of Exercise Set 10

(a) Let $A \in \mathbb{R}^{m\times n}$ with $m \geq n$ and $\mathbf{b} \in \mathbb{R}^m$. If A has rank smaller than n then $\min_{\mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|_2$ has infinitely many solutions.

◯ True ◯ False

(b) Let $R \in \mathbb{R}^{n\times n}$ is upper triangular then

$$\|R^{-1}\|_2 \leq n \cdot \max\{|r_{11}|^{-1}, \ldots, |r_{nn}|^{-1}\}.$$

In particular, this means that $\|R^{-1}\|_2$ can only be large when R has small diagonal entries.

◯ True ◯ False

(c) Consider the problem $\min_{\mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|_1$ with $A \in \mathbb{R}^{m\times n}$, $m \geq n$, and $\mathbf{b} \in \mathbb{R}^m$. Then there is always a minimizer $\mathbf{x}^\star$ such that the residual $A\mathbf{x}^\star - \mathbf{b}$ has at least one zero entry.

◯ True ◯ False

Example 6.10

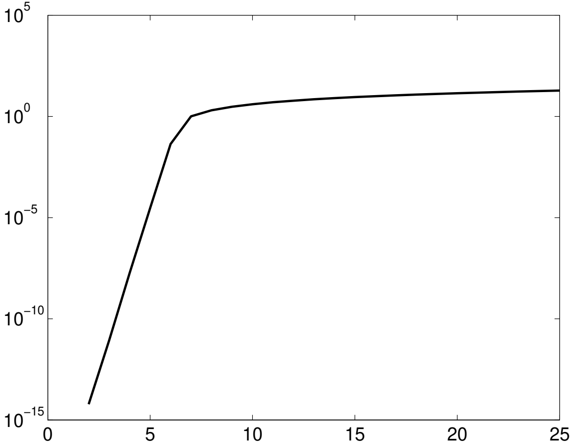
$25 \times n$ matrix (for $n \leq 25$):

$$A = \begin{pmatrix} 1 & t_1^1 & \cdots & t_1^n \\ 1 & t_2^1 & \cdots & t_2^n \\ \vdots & \vdots & & \vdots \\ 1 & t_{25}^1 & \cdots & t_{25}^n, \end{pmatrix}, \qquad t_i = (i-1)/24.$$

Matrix involved when least-squares fitting polynomial of degree n to uniformly sampled data in $[0, 1]$.

Example 6.10

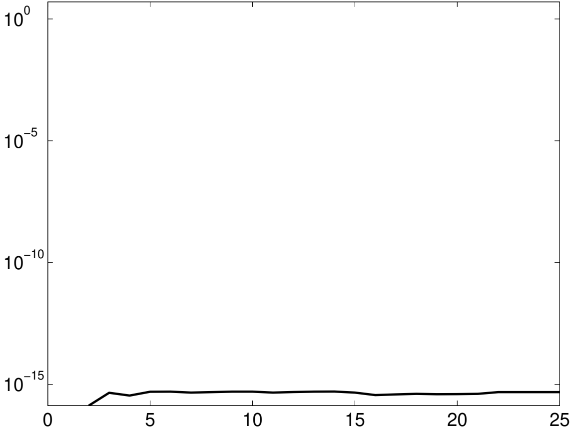
Numerical orthogonality of *computed* orthonormal basis $\widehat{Q}_1$ measured by $\| I_n - \widehat{Q}_1^T \widehat{Q}_1 \|_2$.



Numerical orthogonality vs. *n* for Gram-Schmidt

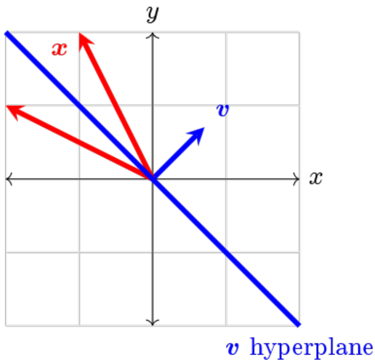
Example 6.10

Numerical orthogonality of *computed* orthonormal basis $\widehat{Q}_1$ measured by $\| I_n - \widehat{Q}_1^T \widehat{Q}_1 \|_2$.



Numerical orthogonality vs. n for
Gram-Schmidt + 1 step of reorthogonalization

Householder reflector



Given $\mathbf{v} \neq 0$, the Householder reflector

$$Q := I_m - \frac{2}{\mathbf{v}^T\mathbf{v}}\mathbf{v}\mathbf{v}^T$$

applied to a vector $\mathbf{x}$ *reflects* $\mathbf{x}$ *at the* $(m-1)$*-dimensional hyperplane* $\mathbf{v}^\perp$.

Householder reflector

Theorem (Properties of Householder reflectors)

Let $\mathbf{0} \neq \mathbf{v} \in \mathbb{R}^m$*. Then the* **Householder reflector**

$$Q := I_m - \frac{2}{\mathbf{v}^T\mathbf{v}}\mathbf{v}\mathbf{v}^T$$

has the following properties:

1. *Q is symmetric,*
2. *Q is orthogonal,*
3. $Q^2 = I_m$.

Householder reflector

Lemma

Let $\mathbf{0} \neq \mathbf{a} \in \mathbb{R}^m$ *and let* $\mathbf{e}_1 \in \mathbb{R}^m$ *denote the first unit vector. Let*

$$\alpha = \|\mathbf{a}\|_2 \quad \text{or} \quad \alpha = -\|\mathbf{a}\|_2.$$

(If $\mathbf{a} = \beta \mathbf{e}_1$ *one uses* $\alpha = -\|\mathbf{a}\|_2 = -|\beta|$*.) Then with* $\mathbf{v} := \mathbf{a} - \alpha \mathbf{e}_1$ *it holds that*

$$Q = I_m - \frac{2}{\mathbf{v}^T\mathbf{v}} \mathbf{v}\mathbf{v}^T \qquad \Rightarrow \qquad Q\mathbf{a} = \alpha \mathbf{e}_1 .$$