Solution 5 – MATH-250 Advanced Numerical Analysis I

The exercise sheet is divided into two sections: quiz and exercises. The quiz will be discussed in the beginning of the lecture on Thursday, March 27. The exercises marked with ($\star$) are graded homework. The exercises marked with **(Python)** are implementation based and can be solved in the Jupyter notebooks which are available on Moodle/Noto. **The deadline for submitting your solutions to the homework is Friday, March 28 at 10h15.**

Quiz

(a) Consider a linear system $A\mathbf{x} = \mathbf{b}$ with a given matrix $A \in \mathbb{R}^{n\times n}$ and a right-hand side $\mathbf{b}$. Which of the following statements are correct?

(i) The linear system has a solution if and only if A is invertible (that is, $\det A \neq 0$).

☐ True ■ False

(ii) If A is not invertible then there is either no solution or infinitely many solutions.

■ True ☐ False

(iii) A random matrix (that is, a matrix with independent normally distributed entries) is invertible with probability 1.

■ True ☐ False

(b) What is the complexity of Gaussian elimination for solving $A\mathbf{x} = \mathbf{b}$?

☐ $O(n)$ ■ $O(n^3)$

☐ $O(n!)$ ☐ $O(n^2)$

(c) Let $\|\cdot\|_p$ denote the ℓ^p norm of a vector for $1 \leq p \leq \infty$. Which of the following statements are correct?

(i) $\|\mathbf{x}\|_1 \leq n\|\mathbf{x}\|_\infty$ and $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_1$ for $\mathbf{x} \in \mathbb{R}^n$

■ True ☐ False

(ii) $\|\mathbf{x}\|_2 \leq \sqrt{n}\|\mathbf{x}\|_1$ and $\|\mathbf{x}\|_1 \leq \|\mathbf{x}\|_2$ for $\mathbf{x} \in \mathbb{R}^n$

☐ True ■ False

(iii) $|\langle \mathbf{x}, \mathbf{y}\rangle| \leq \|\mathbf{x}\|_p\|\mathbf{y}\|_p$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and any $1 \leq p \leq \infty$

☐ True ■ False

(iv) $\| \cdot \|_p$ is not a norm for $p = 1/2$

■ True ☐ False

Solution.

(a) (i) Numerous counter-examples exist. For example

$$A\mathbf{x} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{b}$$

has infinitely many solutions, despite $\det A = 0$.

(ii) If $\det A = 0$, then the kernel of A is non-trivial. Thus, if $\mathbf{x}$ solves $A\mathbf{x} = \mathbf{b}$, then also $A(\mathbf{x} + \mathbf{y}) = \mathbf{b}$ for any $\mathbf{y} \in \ker(A)$, giving rise to infinitely many solutions $\mathbf{x} + \mathbf{y}$.

(iii) The probability that pairwise distinct columns of this matrix are linearly dependent is zero. Hence, it is invertible with probability 1.

(b) See algorithm (there are three nested for-loops).

(c) (i) We can bound

$$\|\mathbf{x}\|_1 = |x_1| + |x_2| + \cdots + |x_n| \leq n \max_{i=1,2,\ldots,n} |x_i| = n\|\mathbf{x}\|_\infty$$

and

$$\|\mathbf{x}\|_\infty = \max_{i=1,2,\ldots,n} |x_i| \leq |x_1| + |x_2| + \cdots + |x_n| = \|\mathbf{x}\|_1.$$

(ii) There exist many counter-examples. For instance $\mathbf{x} = (1, 1)^\top$ has $\|\mathbf{x}\|_1 = 1+1 = 2$ whereas $\|\mathbf{x}\|_2 = \sqrt{1^2 + 1^2} = \sqrt{2}$.

(iii) The inequality often does not hold, for example for $p = \infty$ if $\mathbf{x} = \mathbf{y} = (1, 1)^\top$:

$$|\langle \mathbf{x}, \mathbf{y} \rangle| = |\langle (1, 1)^\top, (1, 1)^\top \rangle| = |1 + 1| = 2$$

but

$$\|\mathbf{x}\|_\infty \|\mathbf{y}\|_\infty = \|(1, 1)^\top\|_\infty \|(1, 1)^\top\|_\infty = 1 \cdot 1 = 1$$

(iv) The triangle inequality does not hold. Consider for instance $\mathbf{x} = (1, 0)^\top$ and $\mathbf{y} = (0, 1)^\top$:

$$\|\mathbf{x} + \mathbf{y}\|_{1/2} = \|(1, 1)^\top\|_{1/2} = (\sqrt{1} + \sqrt{1})^2 = 4$$

but

$$\|\mathbf{x}\|_{1/2} + \|\mathbf{y}\|_{1/2} = \|(1, 0)^\top\|_{1/2} + \|(0, 1)^\top\|_{1/2} = 1 + 1 = 2$$

Exercises

Problem 1. The goal of this exercise is to prove Theorem 2.12; the three-term recurrence relation for the Legendre polynomials defined in the lecture notes:

$$(n+1)q_{n+1}(x) = (2n+1)xq_n(x) - nq_{n-1}(x), \qquad |x| < 1. \tag{1}$$

(a) Using that $q_0, \ldots, q_n$ is an orthogonal basis for $\mathbb{P}_n$, we consider the expansion

$$xq_n = \sum_{i=0}^{n+1} \alpha_i q_i, \quad \alpha_i = \frac{\langle xq_n, q_i \rangle}{\langle q_i, q_i \rangle},$$

where $\langle \cdot, \cdot \rangle$ denotes the L^2 inner product on $[-1, 1]$. Show that $\alpha_i = 0$ except for $i = n-1, n+1$.

(b) Use Theorem 2.11 to determine the leading coefficient of q_n and use this to show that

$$\frac{2n+1}{n+1} xq_n - q_{n+1} \in \mathbb{P}_n$$

(c) Using (b), compute α_{n-1} and α_{n+1}. You may use $\langle q_n, q_n \rangle = \frac{2}{2n+1}$. From this deduce the recurrence relation (1).

Bonus: Prove $\langle q_n, q_n \rangle = \frac{2}{2n+1}$ using Theorem 2.12.

Solution.

(a) Let $i \neq n-1, n, n+1$. Then $\alpha_i = \langle xq_n, q_i \rangle = \langle q_n, xq_i \rangle = 0$ because xq_i is of degree at most $n-1$ and q_n is orthogonal to all such polynomials.

Now, if $i = n$ we have $\alpha_n = \int_{-1}^{1} xq_n(x)^2 dx$. Now, if we show that q_n is even/odd for even/odd n we can show $\alpha_n = 0 \quad \because xq_n^2$ is odd which implies $\alpha_n = 0$.

We show this by induction. For $n = 0$ this follows immediately from that $q_0(x) = 1$ is an odd function. Similarly, for $n = 1$ it is clear that $q_1(x) = x$ is odd.

Hence, now suppose that our hypothesis holds up to some $n \in \mathbb{N}$. Then, if c is some constant we know

$$\begin{aligned} cq_{n+1}(x) &= x^{n+1} - \sum_{i=0}^{n} \frac{\langle x^{n+1}, q_i \rangle}{\langle q_i, q_i \rangle} q_i(x) \\ &= x^{n+1} - \frac{\langle x^{n-1}, q_{n-1} \rangle}{\langle q_{n-1}, q_{n-1} \rangle} q_{n-1}(x) - \frac{\langle x^{n-3}, q_{n-3} \rangle}{\langle q_{n-3}, q_{n-3} \rangle} q_{n-3}(x) - \cdots \end{aligned}$$

Because if $n+1$ is even/odd we know from our inductive assumption $\langle x^{n+1}, q_i \rangle = 0$ for odd/even i. Hence, q_{n+1} is a sum of even/odd functions. Thus, q_{n+1} is even/odd.

(b) From Theorem 2.11 we know that the leading coefficient of q_n is $\frac{(2n)!}{(n!)^2 2^n}$, since the leading coefficient of $\frac{d^n}{dx^n}[(x^2-1)^n]$ is $\frac{(2n)!}{n!}$.

To prove this, we can write: $(x^2-1)^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}x^{2k}$ and hence:

$$\frac{d^n}{dx^n}[(x^2-1)^n] = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\frac{d^n x^{2k}}{dx^n} \tag{2}$$

$$= \sum_{k=0}^{n}\underbrace{\binom{n}{k}(-1)^{n-k}(2k)(2k-1)\dots(2k-n+1)}_{a_k}x^{2k-n} \tag{3}$$

It is possible to find the leading coefficient as the one corresponding to the index $k=n$, that is $a_n = \binom{n}{n}(2n)(2n-1)\dots(n+1) = \frac{(2n)!}{n!}$

This implies that $\frac{2n+1}{n+1}xq_n - q_{n+1} \in \mathbb{P}_n$ because

$$\frac{2n+1}{n+1}\frac{(2n)!}{(n!)^2 2^n} = \frac{(n+1)(2n+1)!}{(n+1)(n+1)!n!2^n} = \frac{(2n+2)(2n+1)!}{((n+1)!)^2 2^{n+1}} = \frac{(2n+2)!}{((n+1)!)^2 2^{n+1}}$$

(c) The result from (b) implies that $xq_n = \frac{n+1}{2n+1}q_{n+1} + p$ for some $p \in \mathbb{P}_n$. Combining this with the result from (a) and $\langle q_n, q_n\rangle = \frac{2}{2n+1}$ gives

$$\alpha_{n+1} = \frac{\langle xq_n, q_{n+1}\rangle}{\langle q_{n+1}, q_{n+1}\rangle} = \frac{n+1}{2n+1}\frac{\langle q_{n+1}, q_{n+1}\rangle}{\langle q_{n+1}, q_{n+1}\rangle} + \langle p, q_{n+1}\rangle = \frac{n+1}{2n+1}$$

and

$$\alpha_{n-1} = \frac{\langle xq_n, q_{n-1}\rangle}{\langle q_{n-1}, q_{n-1}\rangle} = \frac{\langle q_n, xq_{n-1}\rangle}{\langle q_{n-1}, q_{n-1}\rangle} = \frac{n}{2n-1}\frac{\langle q_n, q_n\rangle}{\langle q_{n-1}, q_{n-1}\rangle} = \frac{n}{2n-1}\frac{2n-1}{2n+1} = \frac{n}{2n+1}$$

This gives,

$$xq_n = \frac{n+1}{2n+1}q_{n+1} + \frac{n}{2n+1}q_{n-1}$$

which implies

$$(n+1)q_{n+1} = (2n+1)xq_n - nq_{n-1}$$

which is the desired three-term recurrence relation.

Problem 2. Using Theorem 2.12, show that any root λ of the Legendre polynomial q_{n+1} is also a root of $\det(\lambda B - A)$ with

$$B = \begin{bmatrix} 1 & & & \\ & 3 & & \\ & & \ddots & \\ & & & 2n+1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & 2 & & \\ & 2 & \ddots & \ddots & \\ & & \ddots & \ddots & n \\ & & & n & 0 \end{bmatrix}$$

Verify this statement for small n (say $n=1,2$) in Python with `scipy.linalg.eigvalsh(A,B)`.

Solution. Define the following vector, depending on x

$$q(x) = \begin{bmatrix} q_0(x) \\ q_1(x) \\ \vdots \\ q_n(x) \end{bmatrix}$$

by the three-term recurrence relation be can see that

$$(n+1)q_{n+1}(x)e_{n+1} + Aq(x) = xBq(x) \tag{4}$$

If λ is a root of q_{n+1} then we see that (4) turns into

$$Aq(\lambda) = \lambda Bq(\lambda) \Leftrightarrow 0 = \lambda Bq(\lambda) - Aq(\lambda) \Leftrightarrow \det(\lambda B - A) = 0$$

The solution to the Python implementation is available in the Jupyter notebook `serie05-sol.ipynb` on Moodle.

Problem 3. (Python) Consider the logarithmic spiral curve, whose x- and y-coordinate at time t is given by

$$\begin{cases} x(t) = \exp(-at)\cos(t), \\ y(t) = \exp(-at)\sin(t); \end{cases} \qquad t \in [0, 8\pi]. \tag{5}$$

The parameter $a > 0$ controls how rapidly the spiral curves inward.

(a) Write a Python function which approximates the length of the logarithmic spiral (5). Specifically, ask Chat-GPT to give you the formula for computing the length of a general, smooth curve. Approximate the integral in this formula using the simple Gaussian quadrature with $n = 5$ quadrature points, which is implemented in `scipy.integrate.fixed_quad`/`sp.integrate.fixed_quad`. Use your function to compute the length of the logarithmic spiral (5) for $a = 0.1$ and $a = 0.5$.

(b) Compute the exact length of the logarithmic spiral. Specifically, Ask Chat-GPT to (analytically) compute the length of the logarithmic spiral with the formula it gave you previously. Evaluate the expression for $a = 0.1$ and $a = 0.5$. How far away are they from the approximation?

Warning: It often happens that Chat-GPT gives a wrong answer. Verify that every step in the provided explanation is reasonable, and correct it if necessary.

(c) Visualize the logarithmic spiral for $a = 0.1$ and $a = 0.5$ along with the 5 quadrature nodes of the simple Gaussian quadrature. You can obtain the nodes for the interval $[-1, 1]$ with the function `scipy.special.roots_legendre`/`sp.special.roots_legendre`, but will have to rescale them to the interval $[0, 8\pi]$. Use their locations to explain why the approximation error is significantly larger for one of the values of a.

Solution. Available in the Jupyter notebook `serie05-sol.ipynb` on Moodle.