Solution 4 – MATH-250 Advanced Numerical Analysis I

The exercise sheet is divided into two sections: quiz and exercises. The quiz will be discussed in the beginning of the lecture on Thursday, March 20. The exercises marked with ($\star$) are graded homework. **The deadline for submitting your solutions to the homework is Friday, March 21 at 10h15.**

Quiz

(a) Given a function $f \in C^\infty([a,b])$, we consider the composite trapezoidal rule $Q_h^{(1)}[f]$ and the composite Simpson rule $Q_h^{(2)}[f]$ on the interval $[a,b]$. Which of the following statements are correct?

(i) $\lim_{h\to 0} Q_h^{(1)}[f] = \lim_{h\to 0} Q_h^{(2)}[f] = \int_a^b f(x)\,\mathrm{d}x$.

■ True ☐ False

(ii) $|Q_h^{(1)}[f] - \int_a^b f(x)\,\mathrm{d}x| \le |Q_H^{(1)}[f] - \int_a^b f(x)\,\mathrm{d}x|$ if $h \le H$.

☐ True ■ False

(iii) $|Q_h^{(2)}[f] - \int_a^b f(x)\,\mathrm{d}x| \le |Q_h^{(1)}[f] - \int_a^b f(x)\,\mathrm{d}x|$ for all sufficiently small values $h > 0$.

☐ True ■ False

(iv) If $f(x) \ge 0$ for all $x \in [a,b]$ then

$$0 \le Q_h^{(1)}[f] \le \int_a^b f(x)\,\mathrm{d}x.$$

☐ True ■ False

(v) If f is convex on $[a,b]$ then

$$Q_h^{(1)}[f] \ge \int_a^b f(x)\,\mathrm{d}x.$$

■ True ☐ False

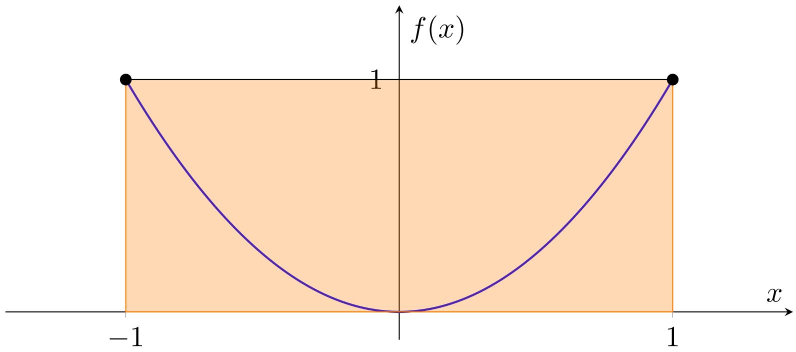
Solution.

(a) (i) See error bounds (Theorem 2.6 in Lecture Notes).

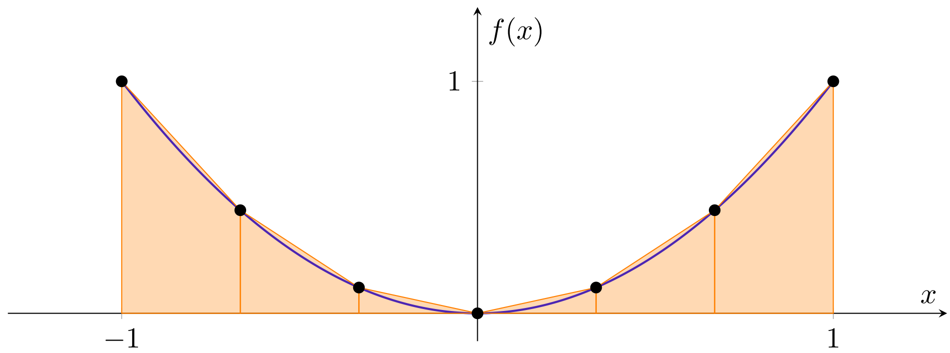
(ii) There are many counter-examples for this. In fact, every function f with $f(-1) = -1$ and $f(1) = 1$ and $\int_{-1}^1 f(x)\ \mathrm{d}x = 0$ but $f(0) \neq 0$ has an exact approximation for $H = 2$, but not for $h = 1$.

(iii) There always exist similar counter-examples as the one in (ii).

(iv) The first inequality always holds. For the second inequality there exist many counter-examples. For example the function $f(x) = x^2$ on the interval $[-1, 1]$ for $h = 2$ violates it:



(v) For a convex function f, any line connecting two function values lies entirely above or on f. Hence, the trapezoidal rule will always over estimate the integral on all subintervals:



Exercises (Exercises marked ($\star$) will be graded.)

Problem 1.

(a) Compute the approximation errors for the trapezoidal rule and Simpson rule applied to the integrals:

$$\int_0^1 x^4 \, \mathrm{d}x \quad \text{and} \quad \int_0^1 x^5 \, \mathrm{d}x$$

(b) Find C such that the trapezoidal rule gives the exact result for the integral

$$\int_0^1 x^5 - Cx^4 \, \mathrm{d}x.$$

(c) Show that the trapezoidal rule gives a better approximation than the Simpson rule in the case that $\frac{15}{14} < C < \frac{85}{74}$.

Solution.

(a) Let $Q^T_{[0,1]}[f]$ and $Q^S_{[0,1]}[f]$ be the approximate integral of any function f defined on $[0,1]$ obtained, respectively, with the trapezoidal rule and Simpson rule. Then

$$\begin{aligned}
\int_0^1 x^4 \mathrm{d}x - Q^T_{[0,1]}\left[x^4\right] &= \frac{1}{5} - \frac{1}{2} = -\frac{3}{10};\\
\int_0^1 x^5 \mathrm{d}x - Q^T_{[0,1]}\left[x^5\right] &= \frac{1}{6} - \frac{1}{2} = -\frac{1}{3};\\
\int_0^1 x^4 \mathrm{d}x - Q^S_{[0,1]}\left[x^4\right] &= \frac{1}{5} - \frac{5}{24} = -\frac{1}{120};\\
\int_0^1 x^5 \mathrm{d}x - Q^S_{[0,1]}\left[x^5\right] &= \frac{1}{6} - \frac{9}{48} = -\frac{1}{48}.
\end{aligned}$$

The errors are then the absolute values of each expression.

(b) Using the results in the previous question and since both the integral and the quadrature rules are linear, one can verify that

$$\int_0^1 \left(x^5 - Cx^4\right) \mathrm{d}x - Q^T_{[0,1]}\left[x^5 - Cx^4\right] = -\frac{1}{3} + \frac{3}{10}C.$$

Therefore, the trapezoidal rule is exact if $\frac{3}{10}C - \frac{1}{3} = 0$, that is, when $C = \frac{10}{9}$.

Similarly, for the Simpson rule

$$\int_0^1 \left(x^5 - Cx^4\right) \mathrm{d}x - Q^S_{[0,1]}\left[x^5 - Cx^4\right] = -\frac{1}{48} + \frac{1}{120}C.$$

Therefore, the Simpson rule is exact if $\frac{1}{120}C - \frac{1}{48} = 0$, that is, when $C = \frac{5}{2}$.

(c) Based on the previous question, the trapezoidal rule gives a better approximation than the Simpson rule when

$$\left|\frac{3}{10}C - \frac{1}{3}\right| < \left|\frac{1}{120}C - \frac{1}{48}\right|.$$

This inequality is satisfied whenever $C_1 < C < C_2$ where

$$\begin{aligned}
\frac{3}{10}C_1 - \frac{1}{3} &= \frac{1}{120}C_1 - \frac{1}{48}\\
\frac{3}{10}C_2 - \frac{1}{3} &= -\left(\frac{1}{120}C_2 - \frac{1}{48}\right).
\end{aligned}$$

That is, $C_1 = \frac{15}{14}$ and $C_2 = \frac{85}{74}$.

Problem 2.

Let $\{b_i\}_{i=1}^N \subset \mathbb{R}$ and $\{c_i\}_{i=1}^N \subset [0,1]$ define a quadrature rule $Q[f] = \sum_{i=1}^N b_i f(c_i)$ with $N \in \mathbb{N}$ nodes for approximating $\int_0^1 f(x)\,\mathrm{d}x$. The quadrature rule Q is called *symmetric* if $c_i = 1 - c_{N+1-i}$ and $b_i = b_{N+1-i}$, for all $i = 1, 2, \ldots, N$.

Show that any symmetric quadrature rule has an even order, that is if Q is exact for polynomials of degree $\leq 2m-2$ for some $m \in \mathbb{N}$, then it is automatically exact for polynomials of degree $2m-1$.

Solution. Let Q be a symmetric quadrature rule on $[0,1]$, and assume that it is exact for all polynomials of degree at most $2m-2$ for some $m \in \mathbb{N}$. Thanks to polynomial division, every polynomial g of degree $2m-1$ can be rewritten as

$$g(t) = C\left(t - \frac{1}{2}\right)^{2m-1} + g_1(t),$$

where $g_1(t)$ has degree $\leq 2m-2$. Since Q integrates exactly g_1 by hypothesis, then it is enough to show that the symmetric quadrature formula Q is exact for $h(t) := \left(t - \frac{1}{2}\right)^{2m-1}$ in order to conclude. Since $h(t)$ is symmetric around the value $\frac{1}{2}$ which is the middle point of the integration interval $[0,1]$, then the exact value of its integral is

$$\int_0^1 h(t)\,\mathrm{d}t = \int_0^1 \left(t - \frac{1}{2}\right)^{2m-1} \mathrm{d}t = 0.$$

Moreover, since Q is symmetric, then for all $i = 1, \ldots, s$,

$$\begin{aligned} b_i h(c_i) + b_{s-1-i} h(c_{s-1-i}) &= b_i\left(c_i - \frac{1}{2}\right)^{2m-1} + b_{s-1-i}\left(c_{s+1-i} - \frac{1}{2}\right)^{2m-1} \\ &= b_i\left(c_i - \frac{1}{2}\right)^{2m-1} + b_i\left(\frac{1}{2} - c_i\right)^{2m-1} = 0. \end{aligned}$$

Therefore, if s is even,

$$Q[h] = \sum_{i=1}^{s} b_i h(c_i) = \sum_{i=1}^{\frac{s}{2}} \left[b_i h(c_i) + b_{s-1-i} h(c_{s-1-i})\right] = 0 = \int_0^1 h(t)\,\mathrm{d}t,$$

and if s is odd, then $c_{\lceil \frac{s}{2} \rceil} = 1 - c_{\lceil \frac{s}{2} \rceil}$ and thus $c_{\lfloor \frac{s}{2} \rfloor + 1} = c_{\lceil \frac{s}{2} \rceil} = \frac{1}{2}$. Therefore, from the same previous reasoning and since $h\left(\frac{1}{2}\right) = 0$, then

$$Q[h] = \sum_{i=1}^{\lfloor \frac{s}{2} \rfloor} \left[b_i h(c_i) + b_{s-1-i} h(c_{s-1-i})\right] + b_{\lceil \frac{s}{2} \rceil} h\left(\frac{1}{2}\right) = 0 = \int_0^1 h(t)\,\mathrm{d}t.$$

Problem 3.

We wish to approximate

$$I_{[a,b]}[f] = \int_a^b f(x)\,\mathrm{d}x$$

using the midpoint rule, where $f \in C^2([a,b])$.

Let $Q_{[a,b]}[f]$ denote the midpoint rule. We know from Theorem 2.5 that

$$|I_{[a,b]}[f] - Q_{[a,b]}[f]| \leq \frac{(b-a)^3}{24} \|f''\|_\infty$$

where $\|f''\|_\infty = \sup_{x\in[a,b]} |f''(x)|$. Let $x_i = a + ih$ for $i = 0, 1, \dots, N$ and $h = \frac{b-a}{N}$. Let $Q_h(f)$ denote the composite midpoint rule, where we apply the midpoint rule to each of the N subintervals of length h.

(a) Derive an upper bound for the approximation error of the composite midpoint rule:

$$|Q_h[f] - I_{[a,b]}[f]| \tag{1}$$

(b) Let $[a, b] = [0, 1]$ and $f(x) = \exp(-x^2)$ and fix $\varepsilon > 0$. Using your result from (a), find the value for N such that (1) is guaranteed to be smaller than ε.

(c) Implement a Python function `midpoint_rule(f, a, b, N)` for the composite midpoint rule. With the functions $f_1(x) = \sqrt{x}$ and $f_2(x) = \frac{1}{\sqrt{x}}$ in $[a, b] = [0, 1]$ display $|Q_h[f] - I_{[a,b]}[f]|$ with respect to N or h on a doubly logarithmic plot (use the `matplotlib` function `matplotlib.pyplot.loglog/plt.loglog`). Using these plots, find suitable values of p that describe the asymptotic behavior $\mathcal{O}(N^{-p})$ of the error for f_1 and f_2, respectively.

Solution.

(a) The composite quadrature rule is given by

$$Q_h[f] = \sum_{i=0}^{N-1} Q_{[x_i,x_{i+1}]}[f]$$

and we also have

$$I_{[a,b]}[f] = \sum_{i=0}^{N-1} I_{[x_i,x_{i+1}]}[f]$$

Hence, the error is

$$\begin{aligned}
|Q_h[f] - I_{[a,b]}[f]| &= \left| \sum_{i=0}^{N-1} Q_{[x_i,x_{i+1}]}[f] - I_{[x_i,x_{i+1}]} \right| \\
&\le \sum_{i=0}^{N-1} |Q_{[x_i,x_{i+1}]}[f] - I_{[x_i,x_{i+1}]}[f]| \\
&\le \sum_{i=0}^{N-1} \frac{(x_{i+1} - x_i)^3}{24} \sup_{x\in[x_i,x_{i+1}]} |f''(x)| \\
&\le \frac{1}{24} \|f''\|_\infty \sum_{i=0}^{N-1} (x_{i+1} - x_i)^3 \\
&= N \frac{(b-a)^3}{24N^3} \|f''\|_\infty = \frac{(b-a)^3}{24N^2} \|f''\|_\infty
\end{aligned}$$

(b) Let $E(N) = |Q_h[f] - I_{[a,b]}[f]|$. By differentiating $f(x) = e^{-x^2}$ twice we get $f''(x) = (4x^2 - 2)e^{-x^2}$. We also have $f'''(x) = -(8x^3 - 12x)e^{-x^2} \ge 0 \quad \forall x \in [0, 1]$. Hence, the maximum of $|f''|$ is attained at either $x = 0$ or $x = 1$. By checking both endpoints we see that $|f''(0)| = 2$ is the maximum value. Hence, $\|f''\|_\infty = 2$. Thus, the upper bound guaranteed by the result in (a) gives

$$E(N) \leq \frac{1}{12N^2} < \varepsilon \Rightarrow N > \frac{1}{2\sqrt{3\varepsilon}}$$

Hence, $N = \lceil \frac{1}{2\sqrt{3\varepsilon}} \rceil$ would suffice.

(c) Available in the Jupyter notebook `serie04-sol.ipynb` on Moodle.