SOLUTION 3 – MATH-250 Numerical Analysis

The exercise sheet is divided into two sections: quiz and exercises. The quiz will be discussed in the beginning of the lecture on Thursday, March 13. The exercises marked with ($\star$) are graded homework. **The deadline for submitting your solutions to the homework is Friday, March 14 at 10h15.**

Quiz

(a) Given interpolation points $x_0, x_1, \ldots, x_n \in [a, b]$, consider the operator $I : C^0([a, b]) \to \Pi_n$, $I : f \mapsto p_n$, which returns the polynomial p_n interpolating a given function f at the interpolation points. Which of the following statements are true about I?

(i) I is a linear operator

■ True ☐ False

(ii) I is surjective

■ True ☐ False

(iii) $|p_n(x)| \leq \max_{x\in[a,b]} |f(x)|$ for all $x \in [a, b]$

☐ True ■ False

Solution.

(a) (i) Consider any two functions f and g. Then for all $\lambda \in \mathbb{R}$ it holds by the interpolation property

$$I[f + \lambda g](x_i) = (f + \lambda g)(x_i) = f(x_i) + \lambda g(x_i) = I[f](x_i) + \lambda I[g](x_i),\ \forall i = 0, 1, \ldots, n.$$

Consequently, I is linear.

(ii) Because $I[q_n] = q_n$ for any polynomial q_n of degree n.

(iii) There exist countless counter-examples; see e.g. Runge phenomenon (Figure 3.1 in Lecture Notes).

Exercises

Problem 1. Rewrite the following expressions such that numerical cancellation is avoided (or at least reduced).

(a) $(x + 1)^{\frac{1}{4}} - 1$ for $x \approx 0$

(b) $\frac{1-\cos(x)}{\sin(x)}$ for $x \approx 0$

(c) $x^2 - y^2$ for $x \approx y$

Solution.

(a)

$$\begin{aligned}(x+1)^{\frac{1}{4}} - 1 &= \frac{((x+1)^{\frac{1}{4}} - 1)((x+1)^{\frac{1}{4}} + 1)}{(x+1)^{\frac{1}{4}} + 1} \\ &= \frac{(x+1)^{\frac{1}{2}} - 1}{(x+1)^{\frac{1}{4}} + 1} \\ &= \frac{((x+1)^{\frac{1}{2}} - 1)((x+1)^{\frac{1}{2}} + 1)}{((x+1)^{\frac{1}{4}} + 1)((x+1)^{\frac{1}{2}} + 1)} \\ &= \frac{x}{((x+1)^{\frac{1}{4}} + 1)((x+1)^{\frac{1}{2}} + 1)}\end{aligned}$$

(b)

$$\frac{1-\cos(x)}{\sin(x)} = \frac{(1-\cos(x))(1+\cos(x))}{\sin(x)(1+\cos(x))} = \frac{\sin(x)}{1+\cos(x)}$$

(c)

$$x^2 - y^2 = (x+y)(x-y)$$

This does not avoid the deletion, but the result has a smaller relative error, as the following argument shows:
We know from Lemma 1.16 that $fl((x+y)(x-y)) = (x+y)(x-y)(1+\theta_3)$ with $|\theta_3| \leq \gamma_3 = \frac{3u}{1-3u}$, therefore the calculated result has a small relative error.
On the other hand $fl(x^2 - y^2) = x^2(1+\theta_2) - y^2(1+\theta_2')$ and thus

$$\left|\frac{fl(x^2-y^2) - (x^2-y^2)}{x^2-y^2}\right| = \left|\frac{x^2\theta_2 - y^2\theta_2'}{x^2-y^2}\right| \leq \frac{x^2+y^2}{|x^2-y^2|}$$

Hence, we cannot guarantee a small relative error with this computation.

Problem 2.

(a) The midpoint rule *approximates* an integral via

$$\int_a^b f(x)\,\mathrm{d}x \approx f\left(\frac{a+b}{2}\right)(b-a).$$

Let $\mathbb{P} = \bigcup_{n=0}^{\infty} \mathbb{P}_n$ be the set of all polynomials. Find the largest $N \in \mathbb{N}_0$ s.t. $\forall p \in \mathbb{P}$ with $\deg(p) \leq N$ the midpoint rule returns the *exact* result for all $a < b$.

Hint: The monomials $1, x, x^2, x^3, \ldots$ form a basis for $\mathbb{P}$. What is the highest degree of $1, x, x^2, x^3, \ldots$ for which the midpoint rule actually returns the exact *result?*

(b) We now consider the calculation of the integral

$$I = \int_0^1 x^{m-1}(1-x)^{n-1}\,\mathrm{d}x$$

where m and n are two integer values $m, n \geq 1$. Apply the midpoint rule to approximate I. For which values of n and m does this rule return the exact value of I?

Solution.

(a) Using the hint we will compute $\int_a^b 1dx, \int_a^b xdx, \int_a^b x^2dx, \int_a^b x^3dx, \cdots$ until we find an N s.t. the formula does not give an exact value. Since the monomials form a basis for the vector space $\mathbb{P}$, we can conclude that the formula will be exact for all $p \in \mathbb{P}$ s.t. $\deg(p) \leq N$.

$$\begin{aligned} p(x) = 1 &: \int_a^b 1dx = (b-a) = p\left(\frac{a+b}{2}\right)(b-a) \\ p(x) = x &: \int_a^b xdx = \frac{1}{2}(b^2-a^2) = \frac{a+b}{2}(b-a) = p\left(\frac{a+b}{2}\right)(b-a) \\ p(x) = x^2 &: \int_a^b x^2dx = \frac{1}{3}(b^3-a^3) = \frac{(a^2+ab+b^2)}{3})(b-a) \neq p\left(\frac{a+b}{2}\right)(b-a) \end{aligned}$$

Hence, we can conclude that it holds for any $p \in \mathbb{P}_1$, but not for all $p \in \mathbb{P}_2$ and therefore not for all $p \in \mathbb{P}_n$ with $n \geq 2$. Therefore, the midpoint rule will return the exact value for linear functions.

(b) The midpoint rule gives $I = \frac{1}{2^{m+n-2}}$. It follows from (a) that this is exact whenever $(m,n) = (2,1), (1,2)$. Now, let us show that it can never be exact whenever $m, n \geq 2$.

Let us denote $I(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$. Integration by parts yields

$$I(m,n) = 0 + \frac{n-1}{m}\int_0^1 x^m(1-x)^{n-2} = \frac{n-1}{m}I(m+1,n-1)$$

Repeating this $n-1$ times yields

$$I(m,n) = \frac{(n-1)(n-2)\cdots 1}{m(m+1)\cdots(m+n-2)}I(m+n-1,1)$$

Note that $I(m+n-1,1) = \frac{1}{m+n-1}$. Hence,

$$I(m,n) = \frac{(n-1)(n-2)\cdots 1}{m(m+1)\cdots(m+n-2)(m+n-1)} = \frac{(m-1)!(n-1)!}{(m+n-1)!} = \frac{1}{m\binom{m+n-1}{n-1}}$$

Hence, now look for (m,n) such that

$$m\binom{m+n-1}{n-1} = 2^{m+n-2} \tag{1}$$

By looking at the prime factors of the terms in (1) we conclude that m and $\binom{m+n-1}{n-1}$ must be powers of 2. In particular, we have for some $l \in \mathbb{N}_0$ However, $\binom{m+n-1}{n-1}$ can only be a power of 2 if $n-1 = 1$ or $n-1 = m+n-2$ (For a proof of this statement see reference in footnote [1]). Thus, (1) can only hold if $n = 2$ or $m = 1$. We only need to consider the case when $n = 2$, since we have assumed that $m \geq 2$.

If $n = 2$ we have

$$\binom{m+n-1}{n-1} = \frac{(m+1)!}{m!} = m+1$$

However, since $m \geq 2$ is a power of 2, $m + 1$ cannot be a power of 2. Thus, (1) cannot hold if $n = 2$.

Therefore, we can conclude that the midpoint rule is only exact if $(m, n) = (2, 1), (1, 2)$.

Problem 3.

(a) Given the interpolation points $x_0 = 0$, $x_1 = \frac{\pi}{2}$, $x_2 = \pi$, write down the polynomial $p_2 \in \mathbb{P}_2$ in the Lagrange basis that interpolates $f(x) = \sin(x)$ at these points. Compute $\int_0^\pi p_2(x)\,\mathrm{d}x$ and $\int_0^\pi f(x)\,\mathrm{d}x$.

(b) Given the interpolation points $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_2 = 1$, write down the polynomial $p_2 \in \mathbb{P}_2$ in the Lagrange basis that interpolates $f(x) = e^x$ at these points.

Solution.

(a) We use the Lagrange polynomials to write out p_2:

$$\begin{aligned} p_2(x) &= f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \\ &= \sin(0)\frac{(x-\frac{\pi}{2})(x-\pi)}{(0-\frac{\pi}{2})(0-\pi)} + \sin\left(\frac{\pi}{2}\right)\frac{(x-0)(x-\pi)}{(\frac{\pi}{2}-0)(\frac{\pi}{2}-\pi)} + \sin(\pi)\frac{(x-0)(x-\frac{\pi}{2})}{(\pi-0)(\pi-\frac{\pi}{2})} \\ &= -\frac{4}{\pi^2}x(x-\pi) \end{aligned}$$

We also have

$$\begin{aligned} \int_0^\pi \sin(x)dx &= 2 \\ \int_0^\pi p_2(x)dx &= -\frac{4}{\pi^2}\left(\frac{\pi^3}{3} - \frac{\pi^3}{2}\right) = \frac{2}{3}\pi \approx 2.094 \end{aligned}$$

Remark: In this case there is an alteriative way of determining $p_2(x)$ which doesn't involve the use of Lagrange polynomials: Note that $\sin(0) = \sin(\pi) = 0$. So p_2 has roots at $x = 0, \pi$. Thus, $p_2(x) = Cx(x - \pi)$. Now we choose C s.t. $p_2\left(\frac{\pi}{2}\right) = 1$. $C = -\frac{4}{\pi^2}$ is the constant we seek. Thus, $p_2(x) = -\frac{4}{\pi^2}x(x - \pi)$.

[1] https://math.stackexchange.com/questions/2338488/binomial-coefficients-that-are-powers-of-2

(b) We again write down the polynomial in the Lagrange basis

$$
\begin{aligned}
p_2(x) &= f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \\
&= 1\cdot\frac{\left(x-\frac{1}{2}\right)(x-1)}{\left(-\frac{1}{2}\right)(-1)} + \sqrt{e}\cdot\frac{x(x-1)}{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)} + e\cdot\frac{x\left(x-\frac{1}{2}\right)}{(1)\left(\frac{1}{2}\right)} \\
&= 2\left(x-\frac{1}{2}\right)(x-1) - 4\sqrt{e}x(x-1) + 2ex\left(x-\frac{1}{2}\right) \\
&= (2-4\sqrt{e}+2e)x^2 + (-3+4\sqrt{e}-e)x + 1
\end{aligned}
$$

We also have

$$
\begin{aligned}
\int_0^1 e^x dx &= e - 1 \approx 1.718 \\
\int_0^\pi p_2(x)dx &= \frac{2-4\sqrt{e}+2e}{3} + \frac{-3+4\sqrt{e}-e}{2} + 1 = \frac{1}{6} + \frac{2}{3}\sqrt{e} + \frac{1}{6}e \approx 1.719
\end{aligned}
$$

Problem 4.

(a) Consider the function $f(x) = e^{2x}$. Find the quadratic polynomial $p_2(x)$ that interpolates f at $x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1$.

(b) By defining $E_2[f](x) = f(x) - p_2(x)$, we know from Theorem 2.3 that

$$
|E_2[f](x)| \leq \frac{\|\omega_{2+1}\|_\infty}{(2+1)!}\|f^{(2+1)}\|_\infty \quad \forall x \in [0,1]
$$

where for a function $h : [0,1] \mapsto \mathbb{R}$ we define $\|h\|_\infty := \sup_{x\in[0,1]} |h(x)|$. Compare the exact error at $x = \frac{3}{4}$ with the a priori error bound $\frac{\|\omega_{2+1}\|_\infty}{(2+1)!}\|f^{(2+1)}\|_\infty$.

(c) Repeat (a) and (b) for the function $g(x) = \sqrt{x}$, the interpolation points $x_0 = \frac{1}{4}, x_1 = 1, x_2 = 4$, and $x = 2$.

Solution.

(a) Let us write $p_2(x) = c_0 + c_1x + c_2x^2$. The system of equations that we have to solve is the following:

$$
\begin{pmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \mathrm{e} \\ \mathrm{e}^2 \end{pmatrix}
$$

We thus obtain $c_0 = 1$, $c_1 = -3 + 4\mathrm{e} - \mathrm{e}^2$, $c_2 = 2\left(1 - 2\mathrm{e} + \mathrm{e}^2\right)$. Therefore, the obtained interpolating quadratic polynomial is

$$
p_2(x) = 1 + (-3 + 4\mathrm{e} - \mathrm{e}^2)x + 2\left(1 - 2\mathrm{e} + \mathrm{e}^2\right)x^2.
$$

(b) We have

$$\left|E_2[f]\left(\frac{3}{4}\right)\right| = \left|f\left(\frac{3}{4}\right) - p_2\left(\frac{3}{4}\right)\right| = \left|\mathrm{e}^{3/2} - \frac{1}{8}\left(-1 + 6\mathrm{e} + 3\mathrm{e}^2\right)\right| \approx 0.2029.$$

Moreover, for the upper bound, we have

$$\left\|f^{(3)}\right\|_\infty = \left\|8\mathrm{e}^{2x}\right\|_\infty = 8\mathrm{e}^2 \quad \text{in the interval } [0,1],$$
$$\|\omega_3\|_\infty = \left\|x\left(x - \frac{1}{2}\right)(x-1)\right\|_\infty = \frac{1}{12\sqrt{3}} \quad \text{in the interval } [0,1]$$

and therefore,

$$\frac{\|\omega_3\|_\infty}{3!}\left\|f^{(3)}\right\|_\infty = \frac{8\mathrm{e}^2}{3!}\frac{1}{12\sqrt{3}} \approx 0.4740$$

is indeed an upper bound for the error $E_2[f]\left(\frac{3}{4}\right)$, even if it overestimates it by a factor larger than 2.

(c) Following the same reasoning as in question (a) with $g(x) = \sqrt{x}$, we obtain $p_2(x) = \frac{1}{45}\left(-4x^2 + 35x + 14\right)$.

(d) Following the same reasoning as in question (b) with $g(x) = \sqrt{x}$, we obtain

$$|E_2[g]\,(2)| = \left|\sqrt{2} - \frac{68}{45}\right| \approx 0.0969$$

and

$$\frac{\|\omega_3\|_\infty}{3!}\left\|g^{(3)}\right\|_\infty \approx \frac{0.476}{6}12 \approx 0.952$$

since, following Theorem 2.3, the interval that should be used for the infinity norm is $[x_0, x_2] = \left[\frac{1}{4}, 4\right]$. Therefore, we again indeed obtain an upper bound of the error, but which is overestimated this time by a factor nearly equal to 10.