Exercise set 3 – MATH-250 Advanced Numerical Analysis I

The exercise sheet is divided into two sections: quiz and exercises. The quiz will be discussed in the beginning of the lecture on Thursday, March 13. The exercises marked with ($\star$) are graded homework. **The deadline for submitting your solutions to the homework is Friday, March 14 at 10h15.**

Quiz

(a) Given interpolation points $x_0, x_1, \ldots, x_n \in [a, b]$, consider the operator $I : C^0([a, b]) \to \Pi_n$, $I : f \mapsto p_n$, which returns the polynomial p_n interpolating a given function f at the interpolation points. Which of the following statements are true about I?

(i) I is a linear operator

☐ True ☐ False

(ii) I is surjective

☐ True ☐ False

(iii) $|p_n(x)| \leq \max_{x\in[a,b]} |f(x)|$ for all $x \in [a, b]$

☐ True ☐ False

Exercises

Problem 1. Rewrite the following expressions such that numerical cancellation is avoided (or at least reduced).

(a) $(x + 1)^{\frac{1}{4}} - 1$ for $x \approx 0$

(b) $\frac{1-\cos(x)}{\sin(x)}$ for $x \approx 0$

(c) $x^2 - y^2$ for $x \approx y$

Problem 2.

(a) The midpoint rule *approximates* an integral via

$$\int_a^b f(x)\,\mathrm{d}x \approx f\left(\frac{a+b}{2}\right)(b - a).$$

Let $\mathbb{P} = \bigcup_{n=0}^{\infty} \mathbb{P}_n$ be the set of all polynomials. Find the largest $N \in \mathbb{N}_0$ s.t. $\forall p \in \mathbb{P}$ with $\deg(p) \leq N$ the midpoint rule returns the *exact* result for all $a < b$.

Hint: The monomials $1, x, x^2, x^3, \ldots$ form a basis for $\mathbb{P}$. What is the highest degree of $1, x, x^2, x^3, \ldots$ for which the midpoint rule actually returns the exact *result?*

(b) We now consider the calculation of the integral

$$I = \int_0^1 x^{m-1}(1-x)^{n-1}\,\mathrm{d}x$$

where m and n are two integer values $m, n \geq 1$. Apply the midpoint rule to approximate I. For which values of n and m does this rule return the exact value of I?

Problem 3.

(a) Given the interpolation points $x_0 = 0$, $x_1 = \frac{\pi}{2}$, $x_2 = \pi$, write down the polynomial $p_2 \in \mathbb{P}_2$ in the Lagrange basis that interpolates $f(x) = \sin(x)$ at these points. Compute $\int_0^\pi p_2(x)\,\mathrm{d}x$ and $\int_0^\pi f(x)\,\mathrm{d}x$.

(b) Given the interpolation points $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_2 = 1$, write down the polynomial $p_2 \in \mathbb{P}_2$ in the Lagrange basis that interpolates $f(x) = e^x$ at these points.

Problem 4.

(a) Consider the function $f(x) = e^{2x}$. Find the quadratic polynomial $p_2(x)$ that interpolates f at $x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1$.

(b) By defining $E_2[f](x) = f(x) - p_2(x)$, we know from Theorem 2.3 that

$$|E_2[f](x)| \leq \frac{\|\omega_{2+1}\|_\infty}{(2+1)!}\|f^{(2+1)}\|_\infty \quad \forall x \in [0,1]$$

where for a function $h : [0,1] \mapsto \mathbb{R}$ we define $\|h\|_\infty := \sup_{x\in[0,1]} |h(x)|$. Compare the exact error at $x = \frac{3}{4}$ with the a priori error bound $\frac{\|\omega_{2+1}\|_\infty}{(2+1)!}\|f^{(2+1)}\|_\infty$.

(c) Repeat (a) and (b) for the function $g(x) = \sqrt{x}$, the interpolation points $x_0 = \frac{1}{4}, x_1 = 1, x_2 = 4$, and $x = 2$.

($\star$) **Problem 5.** This exercise is concerned with approximating the derivative via the finite difference quotient. Let $f : \mathbb{R} \to \mathbb{R}$ be two times continuously differentiable in a neighborhood of $x \in \mathbb{R}$.

(a) Using the remainder term for the Taylor expansion show that there is a constant C such that

$$\left|\frac{f(x+h) - f(x)}{h} - f'(x)\right| \leq Ch,$$

for $h > 0$ sufficiently small. Note that C will depend on h. Approximate C by $\tilde{C} = \lim_{h\to 0} C$.

(b) Suppose that in floating point arithmetic, the *computed* finite difference quotient takes the form

$$\frac{f(x+h)(1+\delta_1) - f(x)(1+\delta_2)}{h}, \quad |\delta_1|, |\delta_2| \leq u.$$

This is a reasonable assumption when h is a power of two and h is neither too large nor too small, so that the addition and division with h are carried out without roundoff error. Establish an upper bound of the form $c\frac{u}{h}$ on the error between the exact and computed finite difference quotient. Again, note that c will depend on h. Approximate c by $\tilde{c} = \lim_{h\to 0} c$.

(c) Use (a) and (b) to prove the following bound on the error between *computed* finite difference quotient and the derivative (using the approximate constants).

$$\left|\frac{f(x+h)(1+\delta_1) - f(x)(1+\delta_2)}{h} - f'(x)\right| \lesssim \frac{|f''(x)|}{2}h + 2\,|f(x)|\,\frac{u}{h} \tag{1}$$

(d) (Python) On a logarithmic set of axes (`matplotlib.pyplot.loglog/plt.loglog`), plot the error for approximating the derivative of $\cos(x)$ at 1 for 20 logarithmically spaced values of h between 10^{-5} and 10^{-10} (`numpy.logspace/np.logspace`). Include a sketch of the resulting plot in your solution. Identify for which value of h the approximation error is smallest. Compare it to the the value of h that (theoretically) minimizes the bound in (c).

Hint: Use `u = numpy.finfo(np.float64).eps / 2` *for u in your expression for the minimizer of (1).*

Remember to upload a scan `homework03.pdf` of your solutions to the homework corresponding to the homework to the submission panel on Moodle until Friday, March 14 at 10h15. Only your submissions to Moodle will be considered for grading.