Solution 11 – MATH-250 Advanced Numerical Analysis I

There is no quiz this week. The exercises marked with ($\star$) are graded homework. The exercises marked with **(Python)** are implementation based and can be solved in the Jupyter notebooks which are available on Moodle/Noto. **The deadline for submitting your solutions to the homework is Friday, May 23 at 10h15.**

Exercises

Problem 1.
You are given data points $(x_i, y_i)^\top, i = 1, \ldots, 8$

i	1	2	3	4	5	6	7	8
x	0.7	3.3	5.6	7.5	6.4	4.4	0.3	-1.1
y	4.0	4.7	4.0	1.3	-1.1	-3.0	-2.5	1.3

Determine the radius r and the center point $(m_1, m_2)^\top$ of a circle so that the circle describes the points as well as possible. The circular equation is

$$(x - m_1)^2 + (y - m_2)^2 = r^2.$$

By multiplying out and rearranging one obtains

$$2xm_1 + 2ym_2 + (r^2 - m_1^2 - m_2^2) = x^2 + y^2.$$

With the new unknown $c = r^2 - m_1^2 - m_2^2$ you get one linear equation in the three unknowns

$$\mathbf{z} = \begin{pmatrix} m_1 \\ m_2 \\ c \end{pmatrix}.$$

(a) Rephrase the circular equations for $i = 1, \ldots, 8$ as an overdetermined system of linear equations $A\mathbf{z} = \mathbf{b}$ where $A \in \mathbb{R}^{8\times 3}$.

(b) Use the QR factorization to determine $\mathbf{z}$ that minimizes $\|A\mathbf{z} - \mathbf{b}\|_2^2$ (use the Python function `numpy.linalg.qr/np.linalg.qr`).

(c) Plot the points and the circle obtained from (b) in the same graph.

Solution.

(a) The system $A\mathbf{z} = \mathbf{b}$ has the form:

$$\begin{pmatrix} 2x_1 & 2y_1 & 1 \\ 2x_2 & 2y_2 & 1 \\ \vdots & \vdots & \vdots \\ 2x_n & 2y_n & 1 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ c \end{pmatrix} = \begin{pmatrix} x_1^2 + y_1^2 \\ x_2^2 + y_2^2 \\ \vdots \\ x_n^2 + y_n^2 \end{pmatrix}$$

(b)-(c) Available in the Jupyter notebook `serie11-sol.ipynb` on Moodle.

Problem 2.

(a) Let $Q = I_m - \frac{2}{\mathbf{v}^\top \mathbf{v}} \mathbf{v}\mathbf{v}^\top \in \mathbb{R}^{m \times m}$ be a Householder reflector. Show that $Q^2 = I_m$.

(b) Show Lemma 6.13 in the lecture notes. That is, let $\mathbf{a} \in \mathbb{R}^m$ be nonzero and let $\mathbf{e}_1 \in \mathbb{R}^m$ be the first unit vector. Let

$$\alpha = \|\mathbf{a}\|_2 \quad \text{or} \quad \alpha = -\|\mathbf{a}\|_2.$$

Then with $\mathbf{v} := \mathbf{a} - \alpha \mathbf{e}_1$ it holds that

$$Q = I_m - \frac{2}{\mathbf{v}^\top \mathbf{v}} \mathbf{v}\mathbf{v}^\top \Rightarrow Q\mathbf{a} = \alpha \mathbf{e}_1.$$

Solution.

(a) Follows from direct computation:

$$\begin{aligned} Q^2 &= (I_m - \frac{2}{\mathbf{v}^\top \mathbf{v}} \mathbf{v}\mathbf{v}^\top)(I_m - \frac{2}{\mathbf{v}^\top \mathbf{v}} \mathbf{v}\mathbf{v}^\top) \\ &= I_m - \frac{4}{\mathbf{v}^\top \mathbf{v}} \mathbf{v}\mathbf{v}^\top + \frac{4}{\mathbf{v}^\top \mathbf{v}} \mathbf{v}\mathbf{v}^\top = I_m \end{aligned}$$

(b) Follows from direct computation:

$$\begin{aligned} Q\mathbf{a} &= \mathbf{a} - \frac{2}{\mathbf{v}^\top \mathbf{v}} \mathbf{v}(\mathbf{v}^\top \mathbf{a}) \\ &= \mathbf{a} - \frac{2}{\|\mathbf{a}\|_2^2 - 2\alpha a_1 + \alpha^2} (\mathbf{a} - \alpha \mathbf{e}_1)(\|\mathbf{a}\|_2^2 - \alpha a_1) \\ &= \mathbf{a} - \frac{1}{\alpha^2 - \alpha a_1 a_1} (\mathbf{a} - \alpha \mathbf{e}_1)(\alpha^2 - \alpha a_1) \\ &= \mathbf{a} - \frac{1}{\alpha - a_1} (\mathbf{a} - \alpha \mathbf{e}_1)(\alpha - a_1) \\ &= \mathbf{a} - \mathbf{a} + \alpha \mathbf{e}_1 \\ &= \alpha \mathbf{e}_1 \end{aligned}$$

Problem 3.

(a) Prove Lemma 7.5 in the lecture notes. That is, for every $k, \ell = 0, 1, \ldots, n-1$, it holds that

$$\sum_{j=0}^{n-1} \omega_n^{kj} \omega_n^{-\ell j} = \begin{cases} 0 & \text{if} \quad k \neq \ell, \\ n & \text{if} \quad k = \ell. \end{cases} \tag{1}$$

Hint: Geometric series.

(b) Using (a), show that

$$\sum_{k=0}^{n-1} \cos\left(l \frac{(2k+1)\pi}{2n}\right) \cos\left(j \frac{(2k+1)\pi}{2n}\right) = \begin{cases} 0 & \text{if } l \neq j \\ \frac{n}{2} & \text{if } l = j \neq 0 \\ n & \text{if } l = j = 0 \end{cases}$$

Solution.

(a) We have to show that

$$\sum_{j=1}^{n-1} \omega_n^{kj}\omega_n^{-\ell j} = n\delta_{k,\ell},$$

where $\omega_n = \exp(-2\pi i/n)$. This implies

$$\sum_{j=1}^{n-1} \omega_n^{kj}\omega_n^{-\ell j} = \sum_{j=1}^{n-1} \exp(2\pi i j(k-\ell)/n) = \frac{1-\exp(2\pi i(k-\ell)/n)^n}{1-\exp(2\pi i(k-\ell)/n)}.$$

If $k \neq n$, $\exp(2\pi i(k-\ell)/n)^n = \exp(2\pi i(k-\ell)) = 1$ and thus the sum is equal to 0. In the case $k = l$, each summand is equal to 1, which yields the statement.

(b) For notational simplicity, let $x_k = \frac{(2k+1)\pi}{2n}$. Recall the following formula:

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\left(\cos(\alpha-\beta)+\cos(\alpha+\beta)\right)$$

and Euler's identity

$$e^{ix} = \cos(x) + i\sin(x).$$

Hence,

$$\begin{aligned}\sum_{j=0}^{N-1}\cos(\ell x_j)\cos(kx_j) =& \frac{1}{2}\sum_{j=0}^{N-1}\left(\cos\left((\ell+k)\,x_j\right)+\cos\left((\ell-k)\,x_j\right)\right)\\ =&\Re\left[\sum_{j=0}^{N-1}\left(e^{i(\ell+k)x_j}+e^{i(\ell-k)x_j}\right)\right] = \Re\left[\sum_{j=0}^{N-1}\left(e^{i(\ell+k)\frac{(2j+1)\pi}{2N}}+e^{i(\ell-k)\frac{(2j+1)\pi}{2N}}\right)\right]\\ =&\frac{1}{2}\Re\left[e^{i(\ell+k)\frac{\pi}{2N}}\sum_{j=0}^{N-1}e^{i(\ell+k)\frac{j\pi}{N}}\right]+\frac{1}{2}\Re\left[e^{i(\ell-k)\frac{\pi}{2N}}\sum_{j=0}^{N-1}e^{i(\ell-k)\frac{j\pi}{N}}\right].\end{aligned}$$

There are three cases. We treat the separately.

$\ell = \mathbf{k} = \mathbf{0}$

$$\sum_{j=0}^{N-1}\cos(\ell x_j)\cos(kx_j) = \frac{1}{2}\Re\left[\sum_{j=0}^{N-1}1\right]+\frac{1}{2}\Re\left[\sum_{j=0}^{N-1}1\right] = \frac{N}{2}+\frac{N}{2} = N.$$

$\ell = \mathbf{k} \neq \mathbf{0}$ We know that the sum of the first terms of a geometric sequence is

$$\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r},$$

thus

$$\begin{aligned}\sum_{j=0}^{N-1}\cos(\ell x_j)\cos(kx_j) =& \frac{1}{2}\Re\left[e^{i(\ell+k)\frac{\pi}{2N}}\frac{1-e^{i(\ell+k)\pi}}{1-e^{i(\ell+k)\frac{\pi}{N}}}\right]+\frac{1}{2}\Re\left[e^{i(\ell-k)\frac{\pi}{2N}}\sum_{j=0}^{N-1}e^{i(\ell-k)\frac{j\pi}{N}}\right]\\ =&\frac{1}{2}\Re\left[e^{i2k\frac{\pi}{2N}}\frac{1-e^{i2k\pi}}{1-e^{i2k\frac{\pi}{N}}}\right]+\frac{1}{2}\Re\left[\sum_{j=0}^{N-1}1\right].\end{aligned}$$

We observe that

$$\frac{1-e^{i(\ell+k)\pi}}{1-e^{i(\ell+k)\frac{\pi}{N}}}=0$$

since $e^{i2k\pi}=1$ and $e^{i2k\frac{\pi}{N}}\neq 1$ because $\frac{2k}{N}\neq 0 \mod 2$ (in fact $0<k\leq N-1$) we conclude

$$\sum_{j=0}^{N-1}\cos(\ell x_j)\cos(kx_j)=0+\frac{1}{2}N=\frac{N}{2}.$$

$\ell\neq \mathbf{k}$ We set $r:=\ell+k$ and $s:=\ell-k$ by noticing that either r and s are even or they are odd at the same time.

We assume r and s even. If m is even,

$$\sum_{j=0}^{N-1}\cos(mx_j)=\Re\left[\sum_{j=0}^{N-1}e^{imx_j}\right]=\Re\left[\frac{1-e^{im\pi}}{1-e^{im\frac{\pi}{N}}}\right]=0,$$

because $e^{im\pi}=1$ and $e^{im\frac{\pi}{N}}\neq 1$ ($\frac{m}{N}\neq 0 \mod 2$). So

$$\begin{aligned}\sum_{j=0}^{N-1}\cos(\ell x_j)\cos(kx_j)=&\frac{1}{2}\sum_{j=0}^{N-1}\left(\cos\left(\left(\ell+k\right)x_j\right)+\cos\left(\left(\ell-k\right)x_j\right)\right)\\=&\frac{1}{2}\sum_{j=0}^{N-1}\left(\cos\left(rx_j\right)+\cos\left(sx_j\right)\right)=0+0=0.\end{aligned}$$

We assume that r and s are odd. We conclude by noting that if m is odd

$$\cos(mx_j)=-\cos(mx_{N-1-j})\qquad \forall\ j=0,\dots,N-1.$$

Indeed

$$\begin{aligned}mx_j=&\frac{m\pi}{N}j+\frac{m\pi}{2N}=:\alpha\\mx_{N-1-j}=&-\left(\frac{m\pi}{N}j+\frac{m\pi}{2N}\right)+m\pi=-\alpha+m\pi.\end{aligned}$$

So

$$\cos(mx_j)=\cos(\alpha)=\cos(-\alpha)=-\cos(-\alpha+m\pi)=-\cos(mx_{N-1-j}).$$