Exercise set 11 – MATH-250 Advanced Numerical Analysis I

There is no quiz this week. The exercises marked with ($\star$) are graded homework. The exercises marked with **(Python)** are implementation based and can be solved in the Jupyter notebooks which are available on Moodle/Noto. **The deadline for submitting your solutions to the homework is Friday, May 23 at 10h15.**

Exercises

Problem 1.
You are given data points $(x_i, y_i)^\top, i = 1, \ldots, 8$

i	1	2	3	4	5	6	7	8
x	0.7	3.3	5.6	7.5	6.4	4.4	0.3	-1.1
y	4.0	4.7	4.0	1.3	-1.1	-3.0	-2.5	1.3

Determine the radius r and the center point $(m_1, m_2)^\top$ of a circle so that the circle describes the points as well as possible. The circular equation is

$$(x - m_1)^2 + (y - m_2)^2 = r^2.$$

By multiplying out and rearranging one obtains

$$2xm_1 + 2ym_2 + (r^2 - m_1^2 - m_2^2) = x^2 + y^2.$$

With the new unknown $c = r^2 - m_1^2 - m_2^2$ you get one linear equation in the three unknowns

$$\mathbf{z} = \begin{pmatrix} m_1 \\ m_2 \\ c \end{pmatrix}.$$

(a) Rephrase the circular equations for $i = 1, \ldots, 8$ as an overdetermined system of linear equations $A\mathbf{z} = \mathbf{b}$ where $A \in \mathbb{R}^{8\times3}$.

(b) Use the QR factorization to determine $\mathbf{z}$ that minimizes $\|A\mathbf{z} - \mathbf{b}\|_2^2$ (use the Python function `numpy.linalg.qr/np.linalg.qr`).

(c) Plot the points and the circle obtained from (b) in the same graph.

Problem 2.

(a) Let $Q = I_m - \frac{2}{\mathbf{v}^\top\mathbf{v}}\mathbf{v}\mathbf{v}^\top \in \mathbb{R}^{m\times m}$ be a Householder reflector. Show that $Q^2 = I_m$.

(b) Show Lemma 6.13 in the lecture notes. That is, let $\mathbf{a} \in \mathbb{R}^m$ be nonzero and let $\mathbf{e}_1 \in \mathbb{R}^m$ be the first unit vector. Let

$$\alpha = \|\mathbf{a}\|_2 \quad \text{or} \quad \alpha = -\|\mathbf{a}\|_2.$$

Then with $\mathbf{v} := \mathbf{a} - \alpha\mathbf{e}_1$ it holds that

$$Q = I_m - \frac{2}{\mathbf{v}^\top\mathbf{v}}\mathbf{v}\mathbf{v}^\top \Rightarrow Q\mathbf{a} = \alpha\mathbf{e}_1.$$

Problem 3.

(a) Prove Lemma 7.5 in the lecture notes. That is, for every $k, \ell = 0, 1, \dots, n-1$, it holds that

$$\sum_{j=0}^{n-1} \omega_n^{kj} \omega_n^{-\ell j} = \begin{cases} 0 & \text{if} \quad k \neq \ell, \\ n & \text{if} \quad k = \ell. \end{cases} \tag{1}$$

Hint: Geometric series.

(b) Using (a), show that

$$\sum_{k=0}^{n-1} \cos\left(l \frac{(2k+1)\pi}{2n}\right) \cos\left(j \frac{(2k+1)\pi}{2n}\right) = \begin{cases} 0 & \text{if } l \neq j \\ \frac{n}{2} & \text{if } l = j \neq 0 \\ n & \text{if } l = j = 0 \end{cases}$$

($\star$) **Problem 4.** Consider points $(t_i, y_i)_{i=1,\dots,m}$ and the weights $w_i > 0, i = 1, \dots, m$. The goal of this exercise is to find the parameters x_1 and x_2 in $g(t) = x_1 + x_2 t$ such that

$$\sum_{i=1}^{m} w_i (g(t_i) - y_i)^2$$

is minimized.

(a) Show that the minimization problem is solved by

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (A^\top W A)^{-1} A^\top W \mathbf{y} \tag{2}$$

for a certain choice of A, W, and $\mathbf{y}$.

(b) Use (2) to complete the function `weighted_linear_least_squares(t, y, w)` which implements the weighted least squares.

(c) Consider the weight function

$$w(t) = (1 + \exp(-10t))^{-1}. \tag{3}$$

Let the weights $w_i = w(t_i), i = 1, \dots, m$. Use these weights to fit the data given by the arrays `t` and `y` in the Jupyter notebook.

(d) Consider the function $f(t) = \cos(t)$ and the points $t_1 = -1$, $t_2 = 0$, and $t_3 = 1$. Compute the coefficients x_1 and x_2 which minimize

$$\sum_{i=1}^{m} w(t_i)(g(t_i) - f(t_i))^2$$

for the unweighted case $w(t) = 1$ and for the weight function (3). Comment on the result.