Moment Generating Functions

The moment generating function (MGF) $M_X(t) : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ of a scalar random variable X is defined as

$$M_X(t) = \mathbb{E}\left[e^{tX}\right], \qquad t \in \mathbb{R}.$$

It need not be finite for $t \neq 0$. But when it is finite zero, magic happens:

Theorem

Let X and Y be scalar random variable, and assume that $M_X(t) < \infty$ and $M_Y(t) < \infty$ for all $t \in I = (-\epsilon, \epsilon)$ for some $\epsilon > 0$. Then, it holds that

1. *M_X is infinitely differentiable on I*
2. *$\mathbb{E}[|X|^k] < \infty$ and $\mathbb{E}[X^k] = \frac{d^k M_X}{dt^k}(0)$, for all $k \geq 1$.*
3. *$F_X = F_Y$ on $\mathbb{R} \iff M_Y = M_X$ on I.*
4. *if $X \perp\!\!\!\perp Y$, then M_{X+Y} is finite and equal to $M_X M_Y$ on I.*