Probability and Statistics I – Corrections 6

Please note : the **reasoning/justifications** for the steps in your solution are also important (not only the final result).

Exercise 1 (a) (1) $\int_{-\infty}^{\infty} f(x)\,dx = 1 = \int_0^1 (ax+bx^2)\,dx = \frac{a}{2}+\frac{b}{3}$ [int. dens. = 1, eval. int.]

(2) $E[X] = 0.6 = \int_0^1 x(ax+bx^2)\,dx = \frac{a}{3}+\frac{b}{4}$ [problem info., defn. exp., eval. int.]

Then (1) $\Rightarrow a + 2b/3 = 2 \Rightarrow a = 2 - 2b/3$;

substitution in (2) : $(2-2b/3)/3 + b/4 = 0.6 \Rightarrow 1/15 = -b/36 \Rightarrow b = -12/5 = -2.4$;

$\Rightarrow a = 2 - 2(-2.4)/3 \Rightarrow a = 18/5 = 3.6$; thus $\boxed{a = 3.6,\ b = -2.4}$

(b) $P\left(X < \frac{1}{2}\right) = \int_0^{1/2} (3.6x - 2.4x^2)\,dx = (1.8x^2 - 0.8x^3)\Big|_{x=0}^{1/2} = \boxed{0.35}$ [prob. cont. RV]

(c) $E[X^2] = \int_0^1 (3.6x^3 - 2.4x^4)\,dx = 0.9x^4 - 0.48x^5\Big|_{x=0}^{1} = 0.9 - 0.48 = 0.42$ [E(g(X)), eval. int.]

$\Rightarrow Var(X) = E[X^2] - (E[X])^2 = 0.42 - 0.36 = \boxed{0.06}$ [alternative formula Var(X)]

Exercise 2 (a) 1. Let **X** represent the mean rainfall (number of inches)

2. **X** $\sim N(\mu = 40.2,\ \sigma^2 = 8.4^2)$ (according to the problem information)

3. Probability that the average rainfall **X** exceeds 44 inches : $P(X > 44)$

4. $P(X > 44) = P\left(\boxed{\frac{X - 40.2}{8.4}} > \frac{44 - 40.2}{8.4}\right)$ [standardize both sides]

$= P(Z > 0.45) = 1 - \underbrace{P(Z \leq 0.45)}_{\Phi(0.45)} = 1 - 0.6736 = 0.3264 \approx \boxed{0.33}$ [$Z \sim N(0,\,1)$; table, simp.]

(b) 1. Let **Y** be the number of years that the rainfall exceeds 44 inches

2. $Y \sim Bin(n = 7,\ p = 0.33)$

Verification of the 4 conditions :

(i) fixed number of trials : $n = 7$ (according to the problem)
(ii) Bernoulli trials : 2 possibilities (exceed 44/not)
(iii) independent trials : by supposition
(iv) same probability of 'success' (exceed 44) for each year : $p = 0.33$ according to (a)

3. Probability that the number of years (that rainfall exceeds 44) equals 3 : $P(Y = 3)$

4. $P(Y = 3) = \boxed{\binom{7}{3}(0.33)^3(0.67)^4}$ (≈ 0.25)

Exercise 3

1. Let $\underline{\mathbf{X}}$ be the number of inches of rainfall, $\underline{\mathbf{Y}}$ the number of years until $\underline{\mathbf{X}} > 50$

2. $\underline{\mathbf{Y}} \sim Geom(p = P(X > 50) = \underline{\underline{0.0062}})$

Verification of the 4 conditions :

(i) fixed number **of 'success'** : $r = 1$ (according to the problem)
(ii) Bernoulli trials : 2 possibilities (exceed 50/not)
(iii) independent trials : by supposition
[*](iv) same probability of 'success' for each year : $P(\text{number of inches } > 50)$:
$\underline{\underline{p = 0.0062}}$

Suppositions : Rainfall $X \sim N(\mu = 40, \sigma^2 = 4^2)$

[*] Calculation of p :

$$\underline{\underline{p}} = P(X > 50) = P\left(\boxed{\frac{X-40}{4}} > \frac{50-40}{4}\right) \qquad \text{[standardize both sides]}$$

$$= P(Z > 2.5) = 1 - \underbrace{P(Z \leq 2.5)}_{\Phi(2.5)} = 1 - 0.9938 = \underline{\underline{0.0062}} \qquad [Z \sim N(0,1); \text{table, simp.}]$$

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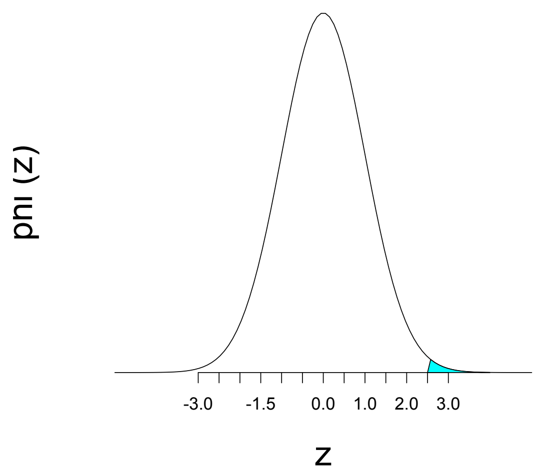
3. Probability that the number of years is more than 10 : $P(Y > 10)$

4. $P(Y > 10) = P(10 \text{ years of 'failure'}) = (1-p)^{10} = (1-0.0062)^{10} = \boxed{(0.9938)^{10}}$ [geom. dist.]



(a) ex. 2a

(b) ex. 3 – calculation of p

Exercise 4 (a) 1. Let $\underline{\mathbf{X}}$ = number of chocolates in the sample

2. $\underline{\mathbf{X}} \sim Hypergeom(n = 5, N = 24, m = 12)$

3. $P(\underline{\mathbf{X}} = 2)$

4. $$= \frac{\binom{12}{2}\binom{12}{3}}{\binom{24}{5}} = \frac{(12!/((2!)(3!)) \times (12!/((3!)(2!))}{24!/((5!)(19!))} = \frac{66 \times 220}{42504} \approx \boxed{0.34}$$

[subst. hypergeom, simp.]

(b) 1. Let $\underline{\mathbf{X}}$ = number of chocolates in the sample

2. $\underline{\mathbf{X}} \sim Hypergeom(n = 5, N = 24, m = 12)$

3. $P(\underline{\mathbf{X}} \leq 2)$

4. $= P(\underline{\mathbf{X}} = 0) + P(X\underline{\mathbf{X}} = 1) + P(\underline{\mathbf{X}} = 2)$ [prob. ME events]

$$= \frac{\binom{12}{0}\binom{12}{5}}{\binom{24}{5}} + \frac{\binom{12}{1}\binom{12}{4}}{\binom{24}{5}} + \frac{\binom{12}{2}\binom{12}{3}}{\binom{24}{5}} = \frac{21252}{42504} = \boxed{0.5}$$ [subst. hypergeom, simp.]

(c) Let $\underline{\mathbf{X}}$ = number of chocolates in the sample

$$E[X] = \sum_{x=0}^{5} x \frac{\binom{12}{x}\binom{12}{5-x}}{\binom{24}{5}} = 0 \cdot P(0) + 1 \cdot P(1) + 2 \cdot P(2) + \cdots + 5 \cdot P(5)$$ [defn. $E[X]$, subst.]

$$= \cdots = \boxed{2.5}$$ [simplification]

(d) Let $\underline{\mathbf{X}}$ = number of chocolates in the sample

Use the **alternative formula** for variance : $Var(X) = E[X^2] - (E[X])^2$

$$E[X^2] = \sum_{x=0}^{5} x^2 \frac{\binom{12}{x}\binom{12}{5-x}}{\binom{24}{5}} = 0^2 \cdot P(0) + 1^2 \cdot P(1) + 2^2 \cdot P(2) + \cdots + 5^2 \cdot P(5)$$

[theorem for $E[g(X)]$, subst.]

$$= \cdots = \quad 7.28$$ [simplification]

$$\Rightarrow \quad Var(X) = 7.28 - (2.5)^2 = 7.28 - 6.25 \approx \boxed{1.03}$$ [alternative formula, simp.]

NOTE : For parts (c) and (d), you could instead use the formulas on slide 18 of the lecture.