SV – Probability and Statistics I

http://moodle.epfl.ch/course/view.php?id=14074

Lecture 2

- Combinatorics
- Introduction – Probability theory

Combinatorics

- In several situations, we desire to have an efficient method to *count* the number of possible results
- In fact, many problems in probability theory can be solved simply *by counting the number of different ways* that a particular *event* can occur
- The mathematical theory of counting : **combinatorics**

Basic counting principle/Principe fondamental de dénombrement

- Suppose that *two* experiments are to be performed
- Then : If *experiment 1* can result in any one of $\underline{m}$ possible outcomes and if, and if *for each outcome of experiment 1*, there are $\underline{n}$ possible outcomes of experiment 2
- $\Rightarrow$ Then there are $\boxed{m \times n}$ possible outcomes of for the two experiments taken together.

Example 2.1 tossing 2 dice :  

Solution : There are ____ possible outcomes for the blue die

and ____ possible outcomes for the red die and therefore

____________ for the 2 experiments taken together

Generalized basic principle/Principe fondamental généralisé

- If r experiments that are to be performed are such that :
  - the first one may result in any of n_1 possible outcomes,
  - if for *each* of these n_1 possible outcomes there are n_2 possible outcomes of the second experiment,
  - if for *each of the possible outcome of the first two experiments* there are n_3 possible outcomes of the third experiment,
  - and so forth ...
- Then there is *a total of* $\boxed{n_1 \times n_2 \times \cdots \times n_r}$ possible outcomes for the r experiments taken together

Examples

Exemple 2.2a How many different license plates (plaques) with 7 characters are possible if the first place is to be occupied by a number, followed by 3 letters then 3 numbers ? ?

$\_\_\_\_ \times \_\_\_\_ \times \_\_\_\_ \times \_\_\_\_ \times \_\_\_\_ \times \_\_\_\_ \times \_\_\_\_$

$(= \_\_\_\_\_\_\_\_)$

Exemple 2.2b How many would be possible if repetition among letters or numbers is not allowed ? ?

$\_\_\_\_ \times \_\_\_\_ \times \_\_\_\_ \times \_\_\_\_ \times \_\_\_\_ \times \_\_\_\_ \times \_\_\_\_$

$(= \_\_\_\_\_\_\_\_)$

Permutations : ordered arrangements

- A **permutation** is an *ordered arrangement* of items
- **Example :** How many different ordered arrangements of the letters a, b, and c are possible ?
  1. *Direct enumeration :* list all of the possibilities and count them
  2. *Basic counting principle :*
     - the first can be any of the 3,
     - the second can then be chosen from the other 2,
     - and the third is 'chosen' from the remaining 1
- By applying the *generalized basic counting principle*, we can see that :
  the number of permutations of *n distinct objects* is $\boxed{n!}$
  (n **factorial** $= n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$)

Example

Example 2.3a 8 rats will be ranked according to their ability to accomplish a task. Supposing that two rats cannot have exactly the same ability, how many rankings are possible ??
[*Hint : does order matter ?*]

Example 2.3b 3 rats are chosen from a group of 9, then placed in 3 cages (C_1, C_2, C_3). How many ways can this be done ??
[*Hint : does order matter ?*]

Permutations : indistinguishable objects

Exemple 2.4 How many different letter arrangements can be formed using the letters $P\ E\ P\ P\ E\ R$?

Solution First, let's count the number of permutations when the 3 P's and 2 E's are *distinguished* from each other $(P_1E_1P_2P_3E_2R) =$ ________

- However, consider any one of these permutations, such as $P_1P_2E_1P_3E_2R$. If we permute the P's among themselves and the E's among themselves, the resulting arrangement still looks like $PPEPER$
  - How many permutations of the P's : ________
    How many permutations of the E's : ________
  - $\Rightarrow$ How many permutations (in total) : ________
- $\Rightarrow$ The number of permutations of n objects, *of which n_1 are indistinguishable, n_2 are indistinguishable, ..., n_r are indistinguishable* :

$$\boxed{\frac{n!}{n_1!n_2!\cdots n_r!}}$$

Combinations : unordered selection

- Consider now that we want to determine the number of groups of r objects that could be formed from a total of n objects (here, *order does not matter*)

Example 2.5 How many different groups of 3 mice could be selected from 5 mice (A, B, C, D, E) ??

Solution We can work this out by **reasoning** as follows :

Since there are ______ ways to choose the first letter, then ______ to choose the next, and ______ to choose the last, then there are __________ ways to select the group of 3 *when order matters*.

However, a given triplet, such as A, B, D, will be counted _________ times.

Thus, the total number of groups that can be formed is _______.

Binomial coefficients

- The expression $\binom{n}{r}$ (*n choose r / r parmi n*), $r \leq n$, is defined as :

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

- This number is also called a *binomial coefficient*
- Any subset of r objects chosen (without replacement/repetition) from a set containing n objects is called a **combination of r objects chosen from among n**
- The number $\binom{n}{r}$ is the number of combinations of r objects chosen from n **when order does not matter**
- This expression represents what *we just figured out using reasoning* based on the Generalized Basic Counting Principle (GBCP) / Principe fondamental de dénombrement généralisé (PFDG)

Examples

Example 2.6 A committee of 3 is to be formed from a group of 20 people. How many different committees are possible ??
[<u>*Hint*</u> *: does order matter ?*]

Example 2.7a From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed ??
[<u>*Hint*</u> *: does order matter ? + GBCP*]

Example 2.7b

Example 2.7b What if 2 of the men refuse to serve on the committee together *??*

[*Hint : does order matter? + GBCP + addition or subtraction*]

Multinomial coefficients

- From the *generalized basic counting principle*, we find that the number of possible divisions of n items into r groups of sizes $n_1, n_2, \ldots, n_r$, respectively, is :

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\ n_2! \cdots\ n_r!}.$$

- This number is called a **multinomial coefficient**

Example 2.8 In a comparative study, 16 people with thyroid illness are to be split into 3 groups of 12, 2 and 2 people.

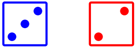
How many ways can this be done *??*

PAUSE

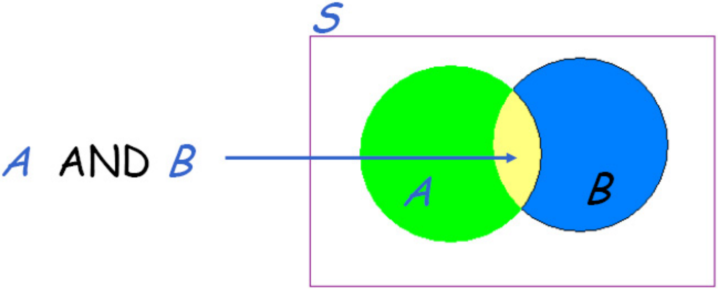
Sample Space

- Let's think about an 'experiment' whose outcome is not predictable with certainty in advance – for example, say one toss of a fair die
- Although the *outcome* will not be known in advance, we might be able to suppose that the set of all *possible* outcomes *is* known
- This set of all possible outcomes is called the **sample space**, denoted by S (or Ω in some books).
- The sample space can be *discrete* or *continuous*
- *Activity : what is the sample space ?*
  - We toss 2 dice : ____________________
  - We toss 2 dice and consider just the *sum* : ____________
  - Survival time after cancer diagnosis : ____________
  - Personal example : ____________________

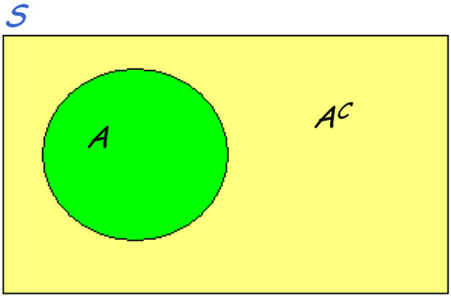
Events

- Any subset $E \subset S$ is called an **event**
- For example :  is an event from the sample space S consisting of the outcomes of tossing 2 dice
- For events E and F of S, the new event $E \cup F$ (the **union** of E and F) consists of all elements that are *in E*, *in F*, **OR** *in both E and F at the same time*
- Similarly, we define the new event $E \cap F$, called the **intersection** of E and F to consist of all outcomes that are *both in E* **and** *in F*
  - When $E \cap F = \varnothing$ (the **empty set**), then E and F are called **mutually exclusive** or **disjoint**
- The new event E^c, referred to as the **complement** of E consists of all outcomes in S that **are not** *in E*

Venn diagram – Union, Intersection



Venn diagram – Complement



Useful rules – event 'algebra'

[NOTE : NOT important for us in this course]

- Commutative laws :
  - $E \cup F = F \cup E$
  - $E \cap F = F \cap E$
- Associative laws :
  - $(E \cup F) \cup G = E \cup (F \cup G)$
  - $(E \cap F) \cap G = E \cap (F \cap G)$
- Distributive laws :
  - $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$
  - $(E \cap F) \cup G = (E \cup G) \cap (F \cup G)$
- DeMorgan's laws :
  - $(\bigcup_{i=1}^{n} E_i)^c = \bigcap_{i=1}^{n} E_i^c$,
  - $(\bigcap_{i=1}^{n} E_i)^c = \bigcup_{i=1}^{n} E_i^c$

Long-run Frequency Interpretation of Probability

- There are several interpretations of the concept of probability but the one we primarily use is the *(long-run) frequency interpretation of probability*
- We consider an experiment with sample space S that is repeatedly carried out under exactly the same conditions every time
- For an event E in S, let $n(E)$ be the *number of times that E occurs* in the first n repetitions of the experiment
- Then $P(E)$, the *probability of the event E* is defined as

$$P(E) = \lim_{n\to\infty} \frac{n(E)}{n}.$$

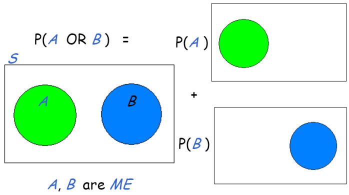
- The probability is thus the **limiting frequency** of occurrence of the event E

Axioms of probability

For each event E in S, we assume that a number $P(E)$ (the **probability of the event E**) is defined and satisfies three *axioms of probability* (that is, statements that we use to characterize probability and that we accept without proof) :

Axioms of probability

1. $0 \leq P(E) \leq 1$
2. $P(S) = 1$
3. For any sequence of *mutually exclusive* events $E_1, E_2, \ldots$, $P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$.



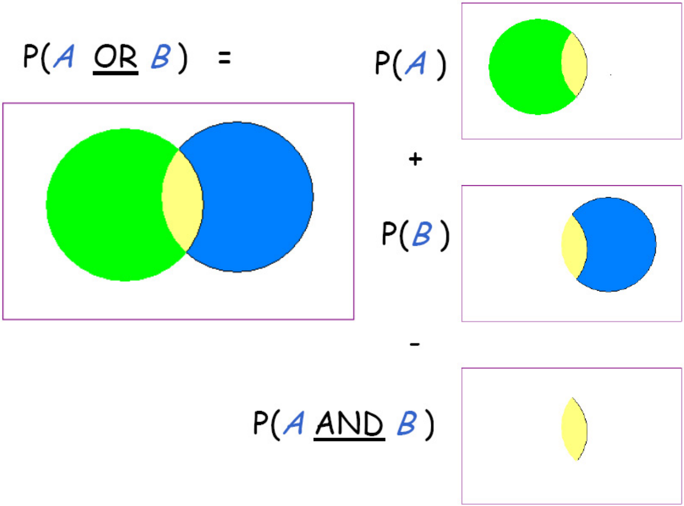
Some simple (but useful) theorems

1. $P(E^c) = 1 - P(E)$
2. If $E \subset F$, then $P(E) \leq P(F)$
3. General addition rule : $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

'Inclusion-exclusion' rule :

$$\begin{aligned}
P(E_1 \cup E_2 \cup \cdots \cup E_n) \quad = \quad & \sum_{i=1}^{n} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} \cap E_{i_2}) \\
& + \sum_{i_1 < i_2 < i_3} P(E_{i_1} \cap E_{i_2} \cap E_{i_3}) - \cdots \\
& + (-1)^{r+1} \sum_{i_1 < i_2 < \cdots < i_r} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_r}) \\
& + \cdots + (-1)^{n+1} P(E_1 \cap E_2 \cap \cdots \cap E_n)
\end{aligned}$$

Venn diagram – Addition/Inclusion-exclusion rule



Equally likely outcomes

- The simplest case : when there is *a finite number of elementary elements* of the sample space, each with *the same probability* of occurence (**equiprobable**)
- Applying Axiom 3, we obtain that for any event E :

$$P(E) = \frac{\text{number of points in } E}{\text{number of points in } S}$$

Exemple 2.9 Suppose that we toss 2 dice, and that each of the 36 possible outcomes is *equally likely*, what is the chance that the sum is 8 *??*

Solution The event $\{\text{sum} = 8\}$ corresponds to the outcomes :

so has *(unconditional)* probability = ________________

Sample space (blue, red)

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Exemple 2.9, cont.

Now I tell you that the first die is ⚂

$\Rightarrow$ *Given* this information, P(sum $= 8$) $=$ *??*

Solution

Our basic calculation of probability when outcomes are *equally likely* :

$$P(E) = \frac{\text{number of points in } E}{\text{number of points in } S}$$

$\Rightarrow$ Now, how many outcomes are there in the sample space S *??*

Given that the first die is ⚂ :

- *the sample space has changed*
- Now, we consider *only the outcomes* for which the first die is ⚂ :

Sample space

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Exemple 2.9, cont

- There are ____________ outcomes in this new sample space
- **AMONG the outcomes in the new sample space**, how many correspond to the event : $\{\text{sum} = 8\}$ *??*

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	**(3,5)**	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

- Therefore, the probability that the sum is 8, GIVEN that the first (blue) is 3 is : ____________