GM – Probabilités et Statistique – Corrections 7

Please note : the **reasoning/justifications** for the steps in your solution are also important (not only the final result).

In class

Exercise 1 (a) $E[\hat{\mu}] = E[\overline{Y}] = E\left[\frac{1}{n}\sum_{i=1}^{n} Y_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[Y_i] = \frac{1}{n}\sum_{i=1}^{n}\mu = \frac{n\mu}{n} = \mu$

$$\Rightarrow\ b(\hat{\mu}) = E[\hat{\mu}] - \mu = \boxed{0 \text{ (unbiased)}}$$

(b) $E[S^2] = \sigma^2 \ \Rightarrow\ E\left[\frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \overline{Y})^2\right] = \sigma^2 \ \Rightarrow\ E\left[\sum_{i=1}^{n}(Y_i - \overline{Y})^2\right] = (n-1)\,\sigma^2$

$$E[\hat{\sigma}^2] = E\left[\frac{1}{n}\sum_{i=1}^{n}(Y_i - \overline{Y})^2\right] = \frac{1}{n}E\left[\sum_{i=1}^{n}(Y_i - \overline{Y})^2\right] = \frac{n-1}{n}\sigma^2$$

$$\Rightarrow\ b(\hat{\sigma}^2) = E[\hat{\sigma}^2] - \sigma^2 = \frac{n-1}{n}\sigma^2 - \sigma^2 = \frac{-1}{n}\sigma^2$$

(c) $b(\hat{\sigma}^2)$ as $n \to \infty = \lim_{n\to\infty} \frac{-\sigma^2}{n} = \boxed{0 \text{ (asymptotically unbiased)}}$

Exercise 2 (a) $L(\lambda) = \prod_{i=1}^{n} e^{-\lambda}\frac{\lambda^{x_i}}{x_i!}\prod_{j=1}^{m} e^{-2\lambda}\frac{(2\lambda)^{y_i}}{y_i!} \quad \propto \quad e^{-(n+2m)\lambda}\lambda^{(n\overline{x}+m\overline{y})}$

(b) $\ell(\lambda) = -(n+2m)\lambda + (n\overline{x} + m\overline{y})\log(\lambda)$

(c) $\frac{d\log\lambda}{d\lambda} = 0 \ \Rightarrow\ -(n+2m) + \frac{(n\overline{x}+m\overline{y})}{\lambda} = 0 \Rightarrow \frac{(n\overline{x}+m\overline{y})}{\lambda} = n+2m$

$$\Rightarrow\ \boxed{\hat{\lambda}_{MLE} = \frac{n\overline{x}+m\overline{y}}{n+2m}}$$

Verify max : $\frac{d^2\ell(\lambda)}{d\lambda^2} = \frac{-(n\overline{x}+m\overline{y})}{\lambda^2} < 0 \ \Rightarrow\ \text{max.}$

(d) $J(\lambda) = \frac{-d^2\ell(\lambda)}{d\lambda^2} = \frac{n\overline{X}+m\overline{Y}}{\lambda^2}$

(e) $I(\lambda) = E[J(\lambda)] = E\left[\frac{n\overline{X}+m\overline{Y}}{\lambda^2}\right] = \frac{1}{\lambda^2}E[n\overline{X}+m\overline{Y}] = \frac{1}{\lambda^2}[n\lambda + 2m\lambda] = \frac{n+2m}{\lambda}$

(f) $E[\hat{\lambda}_{MLE}] = E\left[\frac{n\overline{X}+m\overline{Y}}{n+2m}\right] = \frac{1}{n+2m}(nE[\overline{X}] + mE[\overline{Y}]) = \frac{n\lambda+2m\lambda}{n+2m} = \frac{(n+2m)\lambda}{n+2m} = \lambda$, thus unbiased

(g) 95% IC : $\hat{\lambda}_{MLE} \pm 1.96 \Big/ \sqrt{J(\hat{\lambda}_{MLE})}\,; \qquad J(\hat{\lambda}_{MLE}) = \frac{n\overline{x}+m\overline{y}}{(\hat{\lambda}_{MLE})^2} = \frac{(n+2m)^2}{n\overline{x}+m\overline{y}}$

$$\Rightarrow\ \boxed{\frac{n\overline{x}+m\overline{y}}{n+2m} \pm 1.96 \times \frac{\sqrt{n\overline{x}+m\overline{y}}}{n+2m}} \qquad \text{(you can use 2 instead of 1.96)}$$

At home

Exercise 1 $L(a) = \prod_{i=1}^{n} \frac{a\, b^a}{x_i^{a+1}} = \boxed{\frac{a^n b^{an}}{\prod_{i=1}^n x_i^{a+1}}}$;

$$\boxed{\ell(a) = n \log a + na \log b - (a+1) \sum_{i=1}^{n} \log x_i; \quad 0 < a < \infty}$$

$$\hat{a}_{MLE} : \frac{d\ell(a)}{da} = \frac{n}{a} + n \log b - \sum_{i=1}^{n} \log x_i = 0 \implies \frac{n}{a} = \sum_{i=1}^{n} \log x_i - n \log b$$

$$\implies \boxed{\hat{a}_{MLE} = \frac{n}{\sum_{i=1}^n \log x_i - n \log b}}$$

Verifiy max : $\frac{d^2\ell(a}{da^2} = -\frac{n}{a^2} < 0$, therefore **the extremum ($\hat{a}_{MLE}$) is a maximum**.

Exercise 2 (a) $L(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda y_i} = \boxed{\lambda^n e^{-\lambda \sum_{i=1}^n y_i}}$; $\quad \boxed{\ell(\lambda) = n \log \lambda - \lambda \sum_{i=1}^{n} y_i}$

$$\hat{\lambda}_{MLE} : \frac{d\ell(\lambda)}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} y_i = 0 \implies \frac{n}{\lambda} = \sum_{i=1}^{n} y_i \implies \hat{\lambda}_{MLE} = \frac{n}{\sum_{i=1}^n y_i} = \boxed{1/\bar{y}}$$

(which corresponds to intuition because the mean of the exponential distribution is $1/\lambda$)

Verify max :

$$\frac{d^2\ell(\lambda)}{d\lambda^2} = \frac{d\ell(\lambda)}{d\lambda}\left[\frac{n}{\lambda} - \sum_{i=1}^{n} y_i\right] = -\frac{n}{\lambda^2} < 0, \text{ car } n > 0, \lambda > 0, \text{ donc } \frac{n}{\lambda^2} > 0 \implies -\frac{n\bar{y}}{\lambda^2} < 0,$$

therefore **the extremum ($\hat{\lambda}_{MLE}$) is a maximum**.

(b) $J(\lambda) = \frac{-d^2\ell(\lambda)}{d\lambda^2} = \frac{n}{\lambda^2}$; $\quad I(\lambda) = E[J(p)] = E\left[\frac{n}{\lambda^2}\right] = \frac{n}{\lambda^2}$

(c) 95% IC : $\hat{\lambda}_{MLE} \pm 1.96/\sqrt{J(\hat{\lambda}_{MLE})}$; $J(\hat{\lambda}_{MLE}) = \frac{n}{1/\bar{y}^2} \Rightarrow \boxed{\frac{1}{\bar{y}} \pm 1.96 \times 1/(\bar{y}\sqrt{n})}$

(you can use 2 instead of 1.96)

Exercise 3 Let $Y = Y_1 + \ldots + Y_n$

$$L(p) \propto p^Y (1-p)^{n-Y} \Rightarrow \ell(p) = Y \log(p) + (n-Y) \log(1-p)$$

$$\frac{d\ell(p)}{dp} = 0 \Rightarrow \frac{Y}{p} - \frac{n-Y}{1-p} = 0 \Rightarrow Y(1-p) - (n-Y)p = 0 \Rightarrow \hat{p}_{MLE} = \frac{Y}{n}$$

Verify max : $\frac{d^2\ell(\lambda)}{d\lambda^2} = \frac{-Y}{p^2} - \frac{(n-Y)}{(1-p)^2} < 0 \Rightarrow$ max.

By the *invariance property of the MLE* :

$$\left[\widehat{p(1-p)}\right]_{MLE} = \hat{p}_{MLE}(1 - \hat{p}_{MLE}) = \boxed{\frac{Y}{n}\left(1 - \frac{Y}{n}\right)} \quad \left[= \frac{Y}{n}\left(\frac{n-Y}{n}\right)\right]$$