Exemples 7
(ignorez les nombres ecrits, regardez les nombres en gras et souslignes)

7.1 + 7.2

Ex 10.2 $b(\bar{X}) = E[\bar{X}] - \mu = E\left[\frac{X_1 + \cdots + X_n}{n}\right] = \frac{1}{n} E[X_1 + \cdots + X_n]$

$= \frac{1}{n}[EX_1 + \cdots + EX_n] = \frac{1}{n}[\underbrace{\mu + \cdots + \mu}_{n \text{ fois}}] = \frac{n\mu}{n} = \underline{\mu}$.

$\Rightarrow b(\bar{X}) = \mu - \mu = \underline{0} \Rightarrow \bar{X}$ est non biaisée pour μ

Ex 10.3 $Var(X_1) = \underline{\sigma^2}$

$Var(\bar{X}) = Var\left(\frac{X_1 + \cdots + X_n}{n}\right) = \frac{1}{n^2} Var(X_1 + \cdots + X_n) \overset{X_1, \ldots, X_n \text{ indéps.}}{=} \frac{1}{n^2}[Var X_1 + \cdots + Var X_n]$

$= \frac{1}{n^2}[\underbrace{\sigma^2 + \cdots + \sigma^2}_{n \text{ fois}}] = \frac{n\sigma^2}{n^2} = \underline{\frac{\sigma^2}{n}} \Rightarrow \bar{X}$ est plus efficace (variance est plus petite) que X_1

7.3

Let $X_i \sim Bern(p) \Rightarrow X = \sum_{i=1}^{n} X_i$

$L(p) \propto \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = p^{x}(1-p)^{n-x}$

$\ell(p) = \sum_{i=1}^{n} \left(p^{x_i}(1-p)^{1-x_i}\right) = x \log p + (n-x)\log(1-p)$

$\frac{d\ell}{dp} = \frac{x}{p} - \frac{(n-x)}{1-p} = 0 \Rightarrow \frac{x - \cancel{xp} - np + \cancel{xp}}{p(1-p)} = \frac{(x-np)}{p(1-p)}$

$= 0 \Rightarrow \frac{x}{n} = p \Rightarrow \underline{\underline{\hat{p}_{MLE} = \frac{x}{n}}}$

· Verify max: $\frac{\partial^2 \ell}{\partial p^2} = \frac{-x}{p^2} - \frac{(n-x)}{(1-p)^2} < 0 \Rightarrow \underline{\underline{max}}$

7.4 + 7.5 + 7.6 **(sur les diapos)**

7.7 (parties 1+2+3 ==> regardez ex. 7.3)

4 $J(p) = \frac{-d^2 \ell(p)}{dp^2} = -\left[-\frac{x}{p^2} - \frac{(n-x)}{(1-p)^2}\right] \Rightarrow \frac{x}{(x/n)^2} + \frac{(n-x)}{(1-x/n)^2} = \frac{n\hat{p}}{\hat{p}^2} + \frac{n-n\hat{p}}{(1-\hat{p})^2} = n\left(\frac{1}{\hat{p}} + \frac{1}{(1-\hat{p})}\right) = \frac{n}{\hat{p}(1-\hat{p})}$

5 $I(p) = E\left[\frac{n}{p(1-p)}\right] = \frac{n}{p(1-p)}$

6 an approximate 95% CI for p using the data :

- $n = 10$ (number of Heads $= 9$)

  $9 \pm 2 \times \sqrt{\frac{9/10 \times 1/10}{10}} \approx 0{,}9 \pm 2 \times 0.095$

  (or $0{,}9 - 0.19$ to $0{,}9 + 0.19$)

- $n = 20$ (number of Heads $= 16$)

  $\frac{16}{20} \pm 2 \times \left(\sqrt{\frac{16/20 \times 4/20}{20}}\right) \approx 0.8 \pm 0.09$

  (or $0.8 - 0{,}18$ to $0.8 + 0{,}18$)

- $n = 100$ (number of Heads $= 67$)

  for you...
  $0.67 \pm 2 \times \sqrt{\frac{0.67 \times 0.33}{100}} \approx 0.67 \pm 2 \times 0.047$
  (or $0.67 - 0{,}094$ to $0.67 + 0{,}094$)

7.8

1 $\hat{\lambda}_{EMV}$ supposing that $\sum X_i > 0 \Rightarrow$ see ex. 11.5

2 $\hat{\lambda}_{EMV}$ supposing that $\sum X_i = 0$ If $\sum X_i = 0 \Rightarrow \hat{\lambda} = 0$, BUT for $X \sim \text{Pois}(\lambda)$, $\lambda > 0 \Rightarrow$ MLE does not exist

3 the MLE of $P(X = 0)$ $= \frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda} \Rightarrow$ MLE is $e^{-\bar{x}}$

4 $J(\lambda)$ (see ex. 11.5) $= -\left[\frac{-n\bar{y}}{\lambda^2}\right] = \frac{n\bar{y}}{\lambda^2}$ $\left(\Rightarrow \frac{n\bar{y}}{(\bar{y})^2} = \frac{n}{\bar{y}}\right)$

5 $I(\lambda) = E\left[-\frac{\partial^2 \log f(x|\lambda)}{\partial \lambda^2}\right] = E\left[\frac{X}{\lambda^2}\right] = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$ (so for $X_1 \dots x_n$ $= \frac{n}{\lambda}$)

6 an approximate 95% CI for λ ...

$\bar{x} = \hat{\lambda}_{MLE} \pm 2 \times \sqrt{\bar{x}/n}$