Solution 1

(a) **[seen, 3 points, 1 for each of Ω, $\mathcal{F}$, P]**

To give a *probability space* we must specify Ω, $\mathcal{F}$ and P!

Here $\omega \in \Omega = \{(r, b) : r, b \in \{1, \ldots, 6\}\}$, i.e., there are $|\Omega| = 36$ elementary outcomes of the experiment.

We can suppose that all 2^{36} subsets $\mathcal{B}$ of Ω belong to the event space $\mathcal{F}$, with (by symmetry) $\Pr(\mathcal{B}) = |\mathcal{B}|/36$.

(b) **[unseen, 3 points, 1 for the events and 1 each for the conclusions]**

Here $\mathcal{A} = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$, $\mathcal{B} = \{(2, 1), (2, 2), \ldots, (6, 6)\}$ and $\mathcal{C} = \{(1, 2), (2, 2), \ldots, (6, 6)\}$ (each has 18 elements; they don't need to be written out).

As $\Pr(\mathcal{A}) = 6/36$ and $\Pr(\mathcal{B}) = \Pr(\mathcal{C}) = 1/2$ but they cannot be independent, as

$$\Pr(\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}) = 0 \neq \Pr(\mathcal{A})\Pr(\mathcal{B})\Pr(\mathcal{C}).$$

They are pairwise independent, because

$$\Pr(\mathcal{A}\cap\mathcal{B}) = \Pr(\mathcal{A}\cap\mathcal{C}) = 3/36 = \Pr(\mathcal{A})\Pr(\mathcal{B}) = \Pr(\mathcal{A})\Pr(\mathcal{C}), \quad \Pr(\mathcal{B}\cap\mathcal{C}) = 1/4 = \Pr(\mathcal{B})\Pr(\mathcal{C}).$$

(c) **[seen, 2 points]**

X takes values in $S_X = \{1, 2, 3, 4, 5, 6\}$ with corresponding probabilities

$$f_X(1) = 1/36, \quad f_X(2) = 3/36, \quad f_X(3) = 5/36, \quad f_X(4) = 7/36, \quad f_X(5) = 9/36, \quad f_X(6) = 11/36.$$

(d) **[unseen, 2 points]**

Obviously both expectations are finite (because S_X is finite), and

$$\mathrm{E}(X) = 1 \times \tfrac{1}{36} + 2 \times \tfrac{3}{36} + \cdots + 6 \times \tfrac{11}{36} = 161/36 = 4\tfrac{17}{36},$$

whereas inspection of $\mathcal{A}$ gives

$$\Pr(X = 4 \mid \mathcal{A}) = \Pr(X = 5 \mid \mathcal{A}) = \Pr(X = 6 \mid \mathcal{A}) = \frac{2}{6} = \frac{1}{3},$$

so

$$\mathrm{E}(X \mid \mathcal{A}) = 6\times\Pr(X = 6 \mid \mathcal{A})+5\times\Pr(X = 5 \mid \mathcal{A})+4\times\Pr(X = 4 \mid \mathcal{A}) = 6\times\tfrac{1}{3}+5\times\tfrac{1}{3}+4\times\tfrac{1}{3} = 5.$$

Solution 2

(a) **[seen, 2 points]** X_r is a negative binomial variable, so

$$\Pr(X_r = x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \quad x \in \{r, r+1, \ldots\}, 0 < p < 1.$$

Alternatively a 'bare hands' argument seen in the lectures is that the sequence must be of length x, finishing with a success, and thus with the previous $x-1$ trials having any possible arrangement with $r-1$ successes and $x-1-(r-1) = x-r$ failures. The probability is therefore given by the expression above.

(b) **[seen, 2 points]** $X_1 > x$ iff the first x trials are failures, and this has probability $(1-p)^x$. Hence if $x, u > 0$,

$$\Pr(X_1 > x+u \mid X_1 > u) = \frac{\Pr(X_1 > x+u \cap X_1 > u)}{\Pr(X_1 > u)} = \frac{\Pr(X_1 > x+u)}{\Pr(X_1 > u)} = \frac{(1-p)^{x+u}}{(1-p)^u} = (1-p)^x,$$

which is the lack of memory property of the geometric distribution.

(c) **[seen/unseen, 3 points]** For X_1 and using the sum of a geometric series with $0 < 1-p < 1$, we have

$$\mathrm{E}(X_1) = \sum_{x=1}^{\infty} xp(1-p)^{x-1} = p\frac{\mathrm{d}}{\mathrm{d}p}\sum_{x=1}^{\infty} -(1-p)^x = -p\frac{\mathrm{d}}{\mathrm{d}p}\frac{1-p}{1-(1-p)} = -p\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{1}{p}-1\right) = \frac{1}{p},$$

or perhaps using the result that since $X_1 > 0$, the expectation can also be written as

$$\mathrm{E}(X_1) = \sum_{x=1}^{\infty} \Pr(X_1 \geq x) = \sum_{x=1}^{\infty} (1-p)^{x-1} = \frac{1}{1-(1-p)} = \frac{1}{p}.$$

To compute $\mathrm{E}(X_r)$ we note that X_r has the same distribution as the sum of r independent copies of X_1 (because we wait for the first success, then for the second, then for the third, etc., up to the rth, and the past doesn't matter because of the independence of the individual trials), so $\mathrm{E}(X_r) = r\,\mathrm{E}(X_1) = r/p$, where we have used the linearity of expectation.

(d) **[unseen, 3 points]** We take the hint and condition on the partition :

$$\begin{aligned}
\mathrm{E}(Y) &= \mathrm{E}(Y \mid \mathcal{B}_{11})\Pr(\mathcal{B}_{11}) + \mathrm{E}(Y \mid \mathcal{B}_{10})\Pr(\mathcal{B}_{10}) + \mathrm{E}(Y \mid \mathcal{B}_0)\Pr(\mathcal{B}_0) \\
&= 2 \times p^2 + \mathrm{E}(Y+2) \times p(1-p) + \mathrm{E}(Y+1) \times (1-p) \\
&= (1-p^2)\,\mathrm{E}(Y) + 2p^2 + 2p(1-p) + (1-p) \\
&= (1-p^2)\,\mathrm{E}(Y) + 1 + p,
\end{aligned}$$

solution of which gives $\mathrm{E}(Y) = (1+p)/p^2$; this is plausible, as when $p = 1$ we must have $\mathrm{E}(Y) = 2$. In computing $\mathrm{E}(Y \mid \mathcal{B}_{10})$ we have used the lack of memory property, which implies that after a failure to obtain Y in the first two trials, we start again, but with a lag of 2, giving $\mathrm{E}(Y \mid \mathcal{B}_{10}) = \mathrm{E}(Y+2) = \mathrm{E}(Y) + 2$. Etcetera … **[bonus point for explaining this well]**

Solution 3

(a) **[unseen, 2 points]** The word 'equiprobable' implies use of the uniform distribution, i.e., the length L is uniformly distributed on the interval $(0.9l, 1.1l)$, giving

$$\Pr(L \le x) = \begin{cases} 0, & x \le 0.9l, \\ \frac{x-0.9l}{0.2l}, & 0.9l < x \le 1.1l, \\ 1, & x > 1.1l. \end{cases}$$

(b) **[unseen, 3 points]** The area $A = L^2$ cm^2, so

$$F_A(a) = \Pr(A \le a) = \Pr(L^2 \le a) = \Pr(L \le a^{1/2}) = \begin{cases} 0, & a \le 0.81l^2, \\ \frac{a^{1/2}-0.9l}{0.2l}, & 0.81l^2 < a \le 1.21l^2, \\ 1, & a > 1.21l^2. \end{cases}$$

Hence

$$f_A(a) = F_A'(a) = \begin{cases} \frac{1}{0.4la^{1/2}}, & 0.81l^2 < a \le 1.21l^2, \\ 0, & \text{otherwise.} \end{cases}$$

It's easiest to compute the expectation as

$$\mathrm{E}(A) = \mathrm{E}(L^2) = \int_{0.9l}^{1.1l} x^2 \times \frac{1}{0.2l}\,\mathrm{d}x = l^2 \times \frac{1.1^3 - 0.9^3}{0.6} = 1.003333\, l^2 \ \text{cm}^2,$$

but (correctly) computing this using f_A also gets full marks :

$$\mathrm{E}(A) = \int_{0.81l^2}^{1.21l^2} a \times \frac{1}{0.4la^{1/2}}\,\mathrm{d}a = l^2 \times \frac{1.21^{3/2} - 0.81^{3/2}}{0.6} = 1.003333\, l^2 \ \text{cm}^2.$$

Solution 4 Let $X_1, \dots, X_n$ denote the (independent) arrival times after 8.00.

(a) **[seen, 2 points]** $W > x$ iff all the students arrive after time x, so

$$\Pr(W > x) = \Pr(X_1 > x, \dots, X_n > x) = \prod_{j=1}^{n} \Pr(X_j > x) = e^{-n\lambda x},$$

giving $\Pr(W \le x) = 1 - \exp(-n\lambda x)$ for $x > 0$; i.e., $W \sim \exp(n\lambda)$.

(b) **[unseen, 1 points]** Everyone is on time if all the Xs are smaller than 15, and by independence of the individual arrival times this event has probability $(1 - e^{-15\lambda})^n$.

(c) **[unseen, 2 points]** The number N of students arriving late has a $B(n, p)$ distribution, with $p = e^{-15\lambda}$. Hence (no need for a more explicit expression)

$$\Pr(N \le 3) = \sum_{r=0}^{3} \binom{n}{r} p^r (1-p)^{n-r}.$$

(d) **[seen, 2 points]** The binomial variable N has $\mathrm{E}(N) = np = ne^{-15\lambda}$.

Solution 5 [unseen, generous 3 points] The success probability for strategy (i) is

$$\Pr(\text{A right} \mid \text{A replies})\Pr(\text{A replies}) + \Pr(\text{B right} \mid \text{B replies})\Pr(\text{B replies}) = p \times \tfrac{1}{2} + p \times \tfrac{1}{2} = p,$$

while that for strategy (ii) is

$$\Pr(\text{both right}) + \Pr(\text{coin chooses correctly} \mid \text{disagree})\Pr(\text{disagree}) = p^2 + \tfrac{1}{2} \times 2p(1-p) = p,$$

so both strategies give the same probability of success.