Solution 1

(a) **[3 points, 1 for each of Ω, $\mathcal{F}$, P]** Notez que l'on doit donner un **l'espace de probabilité**, donc Ω, $\mathcal{F}$ et P !
Ici $\Omega = \{(r,g) : r, g \in \{1,\dots,4\}\}$, ainsi $\omega = (r,g)$.
On peut supposer que tous les 2^{16} sous-ensembles B de Ω font partie de la tribu $\mathcal{F}$, avec (par symétrie) $\Pr(B) = |B|/16$ pour tout $B \in \mathcal{F}$.

(b) **[3 points]** On a somme et maximum

$$S(\omega) = r + g, \quad M(\omega) = \max(r,g),$$

dont on répresente les valeurs possible par la table ci-dessous **[1 point]**, dans lequel s/m donne les valeurs de $S(\omega)$ et $M(\omega)$ pour $\omega = (r,g)$.

	g			
r	1	2	3	4
1	2/1	3/2	4/3	5/4
2	3/2	4/2	5/3	6/4
3	4/3	5/3	6/3	7/4
4	5/4	6/4	7/4	8/4

Leur loi conjointe est ainsi donnée par cette table **[1 point]**, qui contient les valeurs de $16 f_{M,S}(m,s)$:

	s						
m	2	3	4	5	6	7	8
1	1	0	0	0	0	0	0
2	0	2	1	0	0	0	0
3	0	0	2	2	1	0	0
4	0	0	0	2	2	2	1

Puisque (par exemple) $\Pr(M=2)\Pr(S=4) = 3/16 \times 3/16 \neq \Pr(M=2, S=4) = 1/16$, S et M ne sont pas indépendantes **[1 point]**.

(c) **[2 points]** La fonction de masse marginale de M est

$$16 f_M(m) = \begin{cases} 1, & m = 1, \\ 3, & m = 2, \\ 5, & m = 3, \\ 7, & m = 4, \\ 0, & \text{sinon.} \end{cases}$$

Donc $\mathrm{E}(M^r) = (1^r \times 1 + 2^r \times 3 + 3^r \times 5 + 4^r \times 7)/16$, donnant $\mathrm{E}(M) = 50/16 = 3\frac{1}{8}$ et $\mathrm{E}(M^2) = 170/16$, et donc $\mathrm{var}(M) = 170/16 - (50/16)^2 \doteq 0.859$.

(d) **[2 points]** La fonction de masse conditionnelle de S sachant que $M = 4$ est **[1 point]**

$$f_S(5 \mid M=4) = f_S(6 \mid M=4) = f_S(7 \mid M=4) = 2/7, \quad f_S(8 \mid M=4) = 1/7,$$

et zero pour $s \notin \{5,6,7,8\}$ **[1 point]**.

Solution 2

(a) **[1 point]** This is clearly $6/9 = 2/3$, since there are 6 ways to choose a safe chocolate.

(b) **[1 point]** This is a conditional probability. If you choose first and survive, there are 8 chocolates left, of which 5 are safe. So her probability of surviving is $5/8$.

(c) **[1 point]** This is a conditional probability. If you choose first and die, there are 8 chocolates left, of which 6 are safe. So her probability of surviving is $6/8 = 3/4$.

(d) **[2 points]** If she chooses first, she may live with probability $2/3$ or die with probability $1/3$. Hence by the law of total probability, your probability of surviving (event S, say) is

$$\begin{aligned}
\Pr(S) &= \Pr(S \mid \text{sister lives})\Pr(\text{sister lives}) + \Pr(S \mid \text{sister dies})\Pr(\text{sister dies}) \\
&= \frac{5}{8} \times \frac{2}{3} + \frac{6}{8} \times \frac{1}{3} \\
&= \frac{16}{24} = \frac{2}{3}.
\end{aligned}$$

Hence it makes no difference which of you goes first.

(e) **[2 points]** Define events A for 'you choose first and survive' and B for 'your sister chooses second and survives' . We want

$$\Pr(A \mid B) = \frac{\Pr(B \mid A)\Pr(A)}{\Pr(B)},$$

using Bayes' theorem. We have $\Pr(A) = 2/3$ (from (a)) and $\Pr(B \mid A) = 5/8$ (from (b)) and $\Pr(B) = 2/3$ (from (d)). Hence $\Pr(A \mid B) = 5/8$: if you know that your sister lived, it makes no difference if you choose first or second.

Solution 3

(a) **[2 points]** Let D_j denote the event that the dog bites on day j and let $C_j = D_1^c \cap \cdots \cap D_j^c$. Then we seek

$$\Pr(D_1^c \cap \cdots \cap D_{n-1}^c \cap D_n) = \Pr(C_{n-1} \cap D_n) = \Pr(C_1) \prod_{j=2}^{n-1} \Pr(D_j^c \mid C_{j-1}) \times \Pr(D_n \mid C_{n-1})$$

and this equals

$$\prod_{j=1}^{n-1} (1 - p_j) \times p_n.$$

(b) **[4 points]** Note that $X = n$ (he is bitten for the first time on day n) corresponds to the event $D_1^c \cap \cdots \cap D_{n-1}^c \cap D_n$, since he must not be bitten on days 1, 2, …, $n-1$ and then he must be bitten on day n. Hence

$$\Pr(X = n) = \Pr(C_{n-1} \cap D_n).$$

If $p_n = (n+1)^{-1}$, then

$$\Pr(X = n) = \Pr(C_{n-1} \cap D_n) = \frac{1}{2} \times \frac{2}{3} \times \cdots \times \frac{n-1}{n} \times \frac{1}{n+1} = \frac{1}{n(n+1)}, \quad n = 1, 2, \ldots.$$

Hence

$$\mathrm{E}(X) = \sum_{n=1}^{\infty} n \Pr(X_A = n) = \sum_{n=1}^{\infty} \frac{n}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty.$$

For the last part, note that

$$\Pr(X > n) = \Pr(C_n) = \prod_{j=1}^{n} (1 - p_j) = \frac{1}{n+1} \to 0, \quad n \to \infty,$$

so $\Pr(X < \infty) = 1$, i.e., there is no probability mass at $+\infty$.

(c) **[2 points]** We seek

$$\mathrm{E}(X \mid X < m) = \sum_{n=1}^{m-1} n \Pr(X = n \mid X < m) = \sum_{n=1}^{m-1} n \frac{1}{n(n+1)} \div \Pr(X \le m-1)$$

and

$$\Pr(X \le m-1) = 1 - \Pr(X > m-1) = 1 - 1/m = \frac{m}{m-1}.$$

Hence

$$\mathrm{E}(X \mid X < m) = \frac{m}{m-1} \sum_{n=1}^{m-1} (n+1)^{-1} = \frac{m}{m-1} \sum_{n=2}^{m} n^{-1} \approx \log m - 1, \quad m \to \infty.$$

Solution 4 Let X_1, X_2, X_3 denote the arrival times after 19.00 of the friends ; then since they arrive independently at random, we can suppose that $X_1, X_2, X_3 \stackrel{\text{iid}}{\sim} U(0,1)$. The arrival times of the first and the last are $U = \min(X_1, X_2, X_3)$ and $V = \max(X_1, X_2, X_3)$.

[2 points] Now as the X_j are independent with distribution function $F(u) = u$ in $0 < u < 1$, we obtain

$$\begin{aligned}
\Pr(U \leq u) &= 1 - \Pr(X_1 > u, X_2 > u, X_3 > u) \\
&= 1 - \Pr(X_1 > u)\Pr(X_2 > u)\Pr(X_3 > u) \\
&= 1 - (1-u)^3, \quad 0 < u < 1,
\end{aligned}$$

and the density is $f_U(u) = 3(1-u)^2$, for $0 < u < 1$.

[2 points] Likewise

$$\Pr(V \leq v) = \Pr(X_1 \leq v)\Pr(X_2 \leq v)\Pr(X_3 \leq v) = v^3, \quad 0 < v < 1,$$

and the density is $f_V(v) = 3v^2$, for $0 < v < 1$.

[3 points] For the final part, we seek $\mathrm{E}(V - U) = \mathrm{E}(V) - \mathrm{E}(U)$, and

$$\mathrm{E}(V) = \int_0^1 v f_V(v)\mathrm{d}v = \tfrac{3}{4}, \quad \mathrm{E}(U) = \int_0^1 u f_u(u)\mathrm{d}u = \int_0^1 3u(1-u)^2\mathrm{d}u = \tfrac{1}{4},$$

using integration by parts. Thus $\mathrm{E}(V - U) = 2/4$, or 30 minutes.

Solution 5 Il y a neuf voyelles e dont trois sont avec accent aigu.

a) **[1 point]** Il y a $\binom{9}{3}$ manières de mettre trois accents aigus sur les neuf e mais une seule manière juste, donc

$$P_1 = \frac{1}{C_9^3} = 1/168.$$

b) **[2 points]** Pour un e donné, avec la même probabilité $1/3$ on peut avoir un accent aigu ou un accent grave ou pas d'accent, donc

$$P_2 = (\tfrac{1}{3})^9.$$

c) **[2 points]** Sans le mot fédérale, il reste que six voyelles e dont une avec accent aigu. Alors les deux probabilités précédentes deviennent

$$P_1 = \binom{6}{1}^{-1} = 1/6, \quad P_2 = (\tfrac{1}{3})^6.$$