Week 14 - Review session

Topology I - point set topology

December 18, 2024

Problem 1. *Give an example of each of the following.*

(a) *A topological space (X, τ) such that every subset $A \subset X$ is both open and closed.*

(b) *A topological space (X, τ) and a subset $A \subset X$ which is neither open nor closed.*

(c) *An open map $f : (X, \tau_X) \to (Y, \tau_Y)$, which is not continuous.*

(d) *A continuous bijection between two topological spaces, which is not a homeomorphism.*

(e) *A simply connected topological space (X, τ) and connected subsets $U, V \in \tau$ such that $U \cup V$ is connected but not $U \cap V$.*

(f) *A complete metric space (M, d) which is not compact (w.r.t. the metric topology).*

(g) *A metric space (M, d) and a subset $A \subset M$ which is both dense and meagre.*

(h) *A topological space (X, τ) which is not metrizable.*

(i) *A metric space (M, d) which is not a Baire space.*

Solution. *In general, there are many possible answers here. I don't remember the examples we discussed during the review session.*

(a) *Any set X endowed with the discrete topology. By definition, any subset is open. Hence, any subset must also be closed.*

(b) *Take any set X with two elements endowed with the indiscrete topology, then any singleton is neither open nor closed.*

(c) *Take any set $X = Y$ with at least two elements, let τ_X be the indiscrete topology, let τ_Y be the discrete topology and let f be the identity.*

(d) *Take any set $X = Y$ with at least two elements, let τ_X be the discrete topology, let τ_Y be the indiscrete topology and let $f : (X, \tau_X) \to (Y, \tau_Y)$ be the identity.*

(e) *Try drawing an example where X, U and V are subsets of $\mathbb{R}^2$.*

(f) *The space $C[0, 1]$ of continuous functions $f : [0, 1] \to \mathbb{R}$ with the sup metric.*

(g) *The standard example is $\mathbb{Q} \subset \mathbb{R}$ (usual metric).*

(h) *Any space that is not Hausdorff. An example that I like is the Zariski topology.*

(i) *$\mathbb{Q}$ with the Euclidean metric.*

Problem 2. *Let X be a set with at least two elements and endow it with the indiscrete topology. Pick two distinct points $p, q \in X$ and consider a sequence (x_n) in X given by $x_n = p$ if n is even, and $x_n = q$ otherwise. Does this sequence converge?*

Solution. *Yes. Any sequence converges in the indiscrete topology (on a set with more than one element). It converges to any point in the space.*

Problem 3. *True or false?*

(a) *If (X, τ) is a topological space which is connected, then we can always find a proper subset $\emptyset \neq A \subsetneq X$ with an empty boundary.*

(b) *Let (X, τ) be a topological space, $A \subset X$ and denote by τ_A the corresponding subspace topology. Then there exists $U \subset A$ such that $U \in \tau$ but $U \notin \tau_A$.*

(c) *If (X, τ) is path connected, then every continuous function $f : (X, \tau) \to (\{0, 1\}, \tau_{disc})$ is constant.*

(d) *Any set endowed with the discrete metric is complete.*

(e) *Any set endowed with the cofinite topology is compact.*

(f) *If (M, d) and (N, d') are two metric space such that (M, τ^d) and $(M, \tau^{d'})$ are homeomorphic, then M is complete if and only if N is complete.*

(g) *If (M, d) is a metric space and $\emptyset \neq A \subset M$ is nowhere dense, then $A \notin \tau^d$.*

Solution.

(a) *False. If we can find such A, then $\emptyset = \partial(A) = cl(A) \setminus int(A)$. Thus, A is both open and closed. In a connected space, this cannot happen for proper subsets.*

(b) *False. If $U \subset A$, then $U \cap A = U$. Thus, $U \in \tau \Rightarrow U \in \tau_A$.*

(c) *True. If (X, τ) is path-connected, it is connected. See Lecture 5.*

(d) *True. All Cauchy-sequences are eventually constant; hence, they converge.*

(e) *True. Take a finite cover $\bigcup U$ of your space X. Pick an open U of this cover. Then U covers all but finitely many points of X. Now, pick finitely many elements of the cover to cover these finitely many points.*

(f) *False. Take $M = \mathbb{R}$ and $N =$ any open interval and the usual metrics.*

(g) *False. Note that if $A \in \tau^d$, then $int(A) \neq \emptyset$. But then $\emptyset \neq int(A) \subset int(cl(A))$, which contradicts the fact that A is nowhere dense.*

Problem 4. *Define the following concepts.*

(a) *Metric topology.* Lecture 2

(b) *Product topology.* Lecture 4

(c) *Connected topological space.* Lecture 5

(d) *Compact topological space.* Lecture 8

(e) *Hausdorff topological space.* Worksheet 4

(f) *Normal topological space.* Lecture 13

(g) *First-countable topological space.* Lecture 3

(h) *The fundamental group of a path-connected topological space.* Lecture 7