# Math 220 - Lecture # 7

October 29, 2024

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**Definition / Proposition** Let $(X, \tau)$ be a topological space, fix $p, q \in X$ and let

$$\mathcal{P}_{p,q} := \{ \gamma : [0,1] \longrightarrow X \, ; \, \gamma \text{ is a path from } p \text{ to } q \}.$$

path - homotopic

① Then $\gamma_1 \sim \gamma_2 \iff \gamma_1 \simeq \gamma_2$ defines an equivalence relation on $\mathcal{P}_{p,q}$ and (exercise)

② When $p = q$, the set of equivalence classes $[\gamma]$

$$\pi_1(X, p) := \mathcal{P}_{p,p} / \sim$$ is called the fundamental group of $X$ at the base point $p$.

**Theorem** ( = Thm 2.31 ) Let $X$ be a topological space and fix $p \in X$. Then $\pi_1(X, p)$ is a group with respect to the binary operation

$$([\gamma_1], [\gamma_2]) \overset{\cdot}{\longmapsto} [\gamma_1] \cdot [\gamma_2] := [\gamma_1 * \gamma_2],$$

called concatenation, where the loop

$\gamma_1 \star \gamma_2 : [0,1] \longrightarrow X$ is given by

(A) $$s \longmapsto \begin{cases} \gamma_1(2s) & \text{if} \quad 0 \leq s \leq 1/2 \\ \gamma_2(2s-1) & \quad\quad 1/2 \leq s \leq 1 \end{cases}.$$

The identity element is the class of the constant path

$$\begin{aligned} e_p : [0,1] &\longrightarrow X \\ s &\longmapsto p \end{aligned}$$

and the inverse of a class $[\gamma]$ is given by the class of the "time-reversed" path.

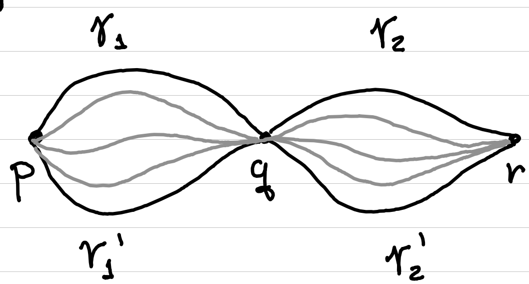
That is, $[\gamma]^{-1} := [\overleftarrow{\gamma}]$, where

(B) $$\begin{aligned} \overleftarrow{\gamma} : [0,1] &\longrightarrow X \\ s &\longmapsto \gamma(1-s). \end{aligned}$$

proof summary

First, note that concatenation of paths as in (A) also makes sense for $\gamma_1 \in \mathcal{P}_{p,q}$ and $\gamma_2 \in \mathcal{P}_{q,r}$. Then $\gamma_1 \star \gamma_2$ defined as in (A) gives a path in $\mathcal{P}_{p,r}$. Similarly, if $\gamma \in \mathcal{P}_{p,q}$, then using (B) we can define $\overleftarrow{\gamma} \in \mathcal{P}_{q,p}$.

Step 1 Concatenation is a well-defined operation. That is, $[\gamma_1 \star \gamma_2]$ only depends on $[\gamma_1]$ and $[\gamma_2]$




If $[\gamma_1] = [\gamma_1']$ and $[\gamma_2] = [\gamma_2']$, then there exist path-homotopies $H^i_t : [0,1] \longrightarrow X$ from $\gamma_i$ to $\gamma_i'$ and

$$H_t : [0,1] \to X$$
$$s \mapsto H(s,t) := H^1_t \star H^2_t \,(s)$$

is a path-homotopy from $\gamma_1 \star \gamma_2$ to $\gamma_1' \star \gamma_2'$.

**Step 2** Associativity follows from the following two claims.

<u>Claim 1</u> Take $\gamma_1 \in \mathcal{P}_{x,y}$, $\gamma_2 \in \mathcal{P}_{y,z}$ and $\gamma_3 \in \mathcal{P}_{z,w}$. Then

- $\gamma_1 \star (\gamma_2 \star \gamma_3)$ is path-homotopic to the path

$$s \overset{\gamma}{\longmapsto} \begin{cases} \gamma_1(2s) & 0 \leq s \leq 1/2 \\ \gamma_2(4s-2) \quad \text{if} & 1/2 \leq s \leq 3/4 \\ \gamma_3(4s-3) & 3/4 \leq s \leq 1 \end{cases}$$

- $(\gamma_1 \star \gamma_2) \star \gamma_3$ is path-homotopic to the path

$$s \overset{\tilde{\gamma}}{\longmapsto} \begin{cases} \gamma_1(4s) & 0 \leq s \leq 1/4 \\ \gamma_2(4s-1) \quad \text{if} & 1/4 \leq s \leq 1/2 \\ \gamma_3(2s-1) & 1/2 \leq s \leq 1 \end{cases}$$

Moreover, $\tilde{\gamma} = \gamma \circ \varphi$, where $\varphi$ is some piece-wise linear function.

<u>Claim 2</u> $[\gamma] = [\tilde{\gamma}]$

**Step 3** If $\gamma \in \mathcal{P}_{p,q}$, then $[e_p]\cdot[\gamma] = [\gamma] = [\gamma]\cdot[e_q]$. Again, it suffices to observe that $e_p \star \gamma$ and $\gamma \star e_q$ can be written as a reparametrization of $\gamma$ by some piece-wise linear function.

**Step 4** Given $\gamma \in \mathcal{P}_{p,q}$, $\gamma \star \overleftarrow{\gamma}$ is path-homotopic to $e_p$ via $H_t \star \overleftarrow{H_t}$ where
$$H_t(s) = \begin{cases} \gamma(s) & \text{if } 0 \leq s \leq 1-t \\ \gamma(1-t) & \text{if } 1-t \leq s \leq 1 \end{cases}$$

$\blacksquare$

**Proposition** (= Prop. 2.33) If $(X, \tau)$ is a path-connected topological space, then $\pi_1(X, p)$ does not depend on the choice of the base point. That is, given $p, q \in X$, $\pi_1(X, p)$ and $\pi_1(X, q)$ are isomorphic (as groups). We then often simply write $\pi_1(X)$ without specifying the base point.

**Definition / Proposition:** (Simply-connected spaces)

A path-connected topological space $(X, \tau)$ is called simply-connected if one (hence all) of the following equivalent conditions holds:

(i) $\pi_1(X)$ is trivial

(ii) Any two paths (in $X$) with the same endpoints are path-homotopic.

**Examples**
- $\mathbb{R}^n$ with the usual topology.
- Any convex subset of $\mathbb{R}^n$.

## The fundamental group of the circle $S^1 \subset \mathbb{R}^2$

**Theorem** $\pi_1(S^1) = \pi_1(S^1, (1,0))$ is an infinite cyclic group generated by the class of the loop $\gamma(s) = (\cos 2\pi s, \sin 2\pi s)$ based @ $(1,0)$.

<u>Remark</u> Let $\gamma_n(s) := \left(\cos 2\pi n s,\ \sin 2\pi n s\right)$.
Then $[\gamma]^n = \underbrace{[\gamma]\cdot \ldots \cdot [\gamma]}_{n \text{ times}} = [\gamma_n]$. Thus,

exercise

we want to prove that every loop based @ $(1,0)$ is path-homotopic to $\gamma_n$ for a unique $n \in \mathbb{Z}$ and $\pi_1(S^1) \simeq \mathbb{Z}$ via $[\gamma_n] \mapsto n$.

**proof of thm** Consider the following map

$$\begin{tikzcd} \mathbb{R} \arrow[r, hook] \arrow[rr, bend left, "p"] & \mathbb{R}^3 \arrow[r, two heads, "q"'] & S^1 \subset \mathbb{R}^2 \end{tikzcd}$$

$$s \mapsto (\cos 2\pi s, \sin 2\pi s, s) \overset{}{\underset{q}{\longmapsto}} (\cos 2\pi s,\ \sin 2\pi s)$$

where $q$ is the restriction of the projection $(x,y,z) \to (x,y)$ to the helix.


Then $\gamma_n = p \circ \tilde{\gamma}_n$, where

$$\tilde{\gamma}_n : I \to \mathbb{R}$$
$$s \mapsto ns$$

Now, $p : \mathbb{R} \to S^1$ is a covering map, meaning that given $x \in S^1$, there exists $\mathcal{U}$ open (in $S^1$) with $x \in \mathcal{U}$ s.t. $p^{-1}(\mathcal{U})$ is a disjoint union of opens (in $\mathbb{R}$), each mapping homeomorphically onto $\mathcal{U}$ via $p$.

In particular,

① 

$$\begin{tikzcd}
 & \mathbb{R} \supset p^{-1}(x) \ni \tilde{x} \arrow[d] \\
{[0,1]} \arrow[ur, dashed, "\substack{\exists!\text{ path } \tilde{\gamma} \\ \text{starting @ } \tilde{x}}"] \arrow[r, "\substack{\text{given } \gamma \text{ a loop} \\ \text{@ } x}"'] & S^1 \ni x = (1,0)
\end{tikzcd}$$

② 

$$\begin{tikzcd}
 & \mathbb{R} \supset p^{-1}(x) \ni \tilde{x} \arrow[d] \\
{[0,1]} \arrow[ur, dashed, "\substack{\exists!\text{ homotopy} \\ \text{of paths from } \tilde{x} \\ \{\tilde{H}_t\}}"] \arrow[r, "\substack{\text{given } \{H_t\} \\ \text{hom. of loops @ } x}"'] & S^1
\end{tikzcd}$$

And we can prove that 1 + 2 imply the theorem.

In fact, if $\alpha : [0,1] \to S^1$ is a loop @ $x = (1,0)$, then by 1 $\exists!$ lift $\tilde{\alpha} : [0,1] \to \mathbb{R}$ starting at $\tilde{x} = 0$. Since $p \circ \tilde{\alpha} = \alpha$ and $p^{-1}(x) = \mathbb{Z}$, we have that $\tilde{\alpha}(1) = n$ for some $n \in \mathbb{Z}$. Now, another path in $\mathbb{R}$ from 0 to n is $\tilde{\gamma}_n$. Composing the linear homotopy $(1-t)\tilde{\alpha} + t\tilde{\gamma}_n$ with $p$ gives that $[\alpha] = [\gamma_n]$.

Finally, if $\alpha \cong \gamma_n$ and $\alpha \cong \gamma_m$, then $\exists$ homotopy of paths $\{H_t\}$ from $\gamma_n$ to $\gamma_m$. By 2 this lifts to a homotopy of paths $\{\tilde{H}_t\}$ starting @ $\tilde{x} = 0$. Uniqueness in 1 implies $\tilde{H}_0 = \tilde{\gamma}_n$ and $\tilde{H}_1 = \gamma_m$ and since $\tilde{H}_t(1)$ is independent of $t$, $n = m$.

∎

**Remark** Given a loop $\alpha : [0,1] \to S^1$ at $x = (1,0)$, the integer $n := \tilde{\alpha}(1) - \tilde{\alpha}(0)$ (where $\tilde{\alpha}$ is a lift of $\alpha$) is called the degree of $\alpha$.