GROUP THEORY 2024 - 25, SOLUTION SHEET 12

Exercise 1. To do yourself. Ask the assistant if something is unclear.

Exercise 2. Denote by g the morphism $G \to F$ in the short exact sequence. Let $A \subset F$ be a set of generators of F, that is $F \cong \langle A \rangle$. For each $a_\alpha \in A$ let $b_\alpha \in G$ be an arbitrary preimage of a_α by g (we can always find at least one such preimage, because g is surjective). Consider the set map $\varphi : A \to G$ given by $a_\alpha \mapsto b_\alpha$. Because F is free with generators in A, φ induces a group homomorphism $\varphi : F \to G$. Clearly, the composition $g \circ \varphi$ fixes all generators of F, and again by the universal property we conclude this map must be the identity on F, i.e. g splits.

Exercise 3. To prove that F is torsion free we observe that any element of F can be written in the form $\alpha\beta\alpha^{-1}$ where β is a cyclically reduced word, i.e. if $\beta = s_1 \cdots s_n$ then $s_1 \neq s_n^{-1}$. For any $m > 0$ we have

$$(\alpha\beta\alpha^{-1})^m = \alpha\beta^m\alpha^{-1}$$

and as β is cyclically reduced no cancellation can happen inside of β^m. Thus if $\alpha\beta\alpha^{-1}$ was non-trivial also $(\alpha\beta\alpha^{-1})^m$ stays non-trivial for any $m > 0$ which proves that F is torsion free.

Now, let $a \in F \setminus \{1\}$ and denote the last letter (element of S) of the reduced word form of a by x. As $|S| \geq 2$, we can pick $y \in S$ different than x. Then it is straightforward to check that $ay \neq ya$, so a is not in the center of F. By arbitrary choice of a we conclude that the center is trivial.

Exercise 4. Consider the set map $\varphi : X \cup Y \to F_X$ given by the identity on X and mapping elements of Y to the empty word. It induces a surjective group homomorphism $\varphi : F_{X\cup Y} \to F_X$. Let us prove that its kernel is the normal subgroup generated by Y to conclude by the first isomorphism theorem.

The normal subgroup generated by Y is obviously contained in $\ker\varphi$ as φ carries the generators coming from Y to the empty word.

Now, let a be in the kernel of φ and write $a = X_1Y_2X_2 \dots X_nY_n$ where X_i are elements of F_X and Y_i are elements of F_Y (may be 1). Then $1 = \varphi(a) = X_1X_2 \dots X_n$ and hence we must have $X_n = X_{n-1}^{-1} \dots X_1^{-1}$. Then

$$a = X_1Y_1 \dots X_{n-1}Y_{n-1}(X_{n-1}^{-1} \dots X_1^{-1})Y_n = X_1(Y_1X_2(\dots)X_2^{-1})X_1^{-1}Y_n$$

which is clearly an element of the normal subgroup generated by Y, so we are done.

Exercise 5. (1) Since $i^2 = j^2 = k^2$, we have that $-1 = i^2$ commutes with the generators i, j and k. Hence $-1 \in Z(Q_8)$.

(2) Using the relations observe that $ij = k$, $jk = i$, $ki = j$, $ij = -ji, jk = -kj, ki = -ki$. Using this any word in i, j and k can be written as an element of the set $\{\pm 1, \pm i, \pm j, \pm k\}$. Therefore $|Q_8| = 8$.

(3) In the last part we showed that $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. Using the identities of products of elements written in the last part it follows that the only central elements are 1 and -1. Hence $Z(Q_8) = \{-1, 1\}$.

(4) If H is a non-trivial and proper subgroup of Q_8 then it has order 2 or 4. Since the index of an order 4 subgroup of Q_8 is 2 it would be normal. Now note that -1 is the only element of Q_8 of order 2. So the only order 2 subgroup of Q_8 is $\{1, -1\}$, which is the centre and hence normal.

Exercise 6. For this exercise, we will use repeatedly proposition 27: the commutator subgroup $[G, G] \lhd G$ is normal in G, and for any other normal subgroup $H \lhd G$ such that the quotient G/H is abelian, we have $[G, G] \lhd H$.

(1) the group A_n is simple and non abelian for all $n \geq 5$, which implies that $[A_n, A_n] = A_n$ and $A_n^{ab} = 1$.
(2) We know that $V_4 \lhd A_4$ is a normal subgroup such that A_4/V_4 is abelian (of order 3). Hence $[A_4, A_4] \lhd V_4$. The commutator subgroup can't be trivial since A_4 is not abelian. It cannot be of order 2 since A_4 doesn't have a normal subgroup of order 2 (because the centre of A_4 has no element of order 2). Hence $[A_4, A_4] = V_4$ and $A_4^{ab} = A_4/V_4 \cong \mathbb{Z}/3\mathbb{Z}$.
(3) We know that $A_n \lhd S_n$ with abelian quotient. Since A_n is simple and S_n is not abelian, we must have that $[S_n, S_n] = A_n$. It follows that $S_n^{ab} = S_n/A_n \cong \mathbb{Z}/2\mathbb{Z}$.
(4) We have seen in the lectures that $F_S^{ab} = F_S/[F_S, F_S] = \mathbb{Z}^S = \mathbb{Z} \oplus \mathbb{Z}$.
(5) From the first relation we observe that $a^2 = b^{-3}$, substituting in the second relation it implies that $1 = a^4 b^5 = (b^{-3})^2 b^5 = b^{-6+5} = b^{-1}$, which means that $b = 1$. It implies that the group admits the presentation $\langle a | a^2 \rangle$ which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. It is its own abelianization.

Exercise 7. We write G for the group given by the presentations of each point.

(1) Since the two generators have order 2, the last relation implies that $ab = (ab)^{-1} = b^{-1}a^{-1} = ba$. This shows that G is abelian with two generators of order 2, i.e. $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
(2) Let $S = \{a, b\}$ and let $f : F_S \to A_4$ be the unique group homomorphism such that $f(a) = (123)$ and $f(b) = (234)$, given by lemma 16. Since A_4 is generated by these two 3-cycles (you can verify this by hand), we have that $F_S/\ker(f) \cong A_4$. Let $N \lhd F_S$ be the normal subgroup generated by $R = \{a^3, b^3, (ab)^2\}$. Since those relations are satisfied by their image by f in A_4, we obtain that $N \subset \ker(f)$. Since by definition $G = F_S/N$, we obtain by the correspondence theorem that $G/\pi(\ker(f)) \cong (F_S/N)/\ker(f)/N) \cong F_S/\ker(f) \cong A_4$. This means that A_4 is a quotient of G. If we show that G contains at most 23 elements, we would have that $|G| = 12$ and $G = A_4$. We propose two solutions to count the number of elements in G.

(a) Using the relations, observe that $a' := ab$ and $b' = ba$ satisfy the relations $a'^2 = b'^2 = (a'b')^2 = 1$. By the previous point, those two elements generate a copy of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in G. Using that $ab = b^2a^2$, we find that

$$\begin{aligned} aa'a^{-1} &= aa'a^2 = aaba^2 = ab^2a^2a^2 = a'b' \in \langle a', b' \rangle \leq G \\ ba'b^{-1} &= bab^3 = b' \in \langle a', b' \rangle \leq G \\ ab'a^{-1} &= a' \in \langle a', b' \rangle \leq G \\ bb'b^{-1} &= b'a' \in \langle a', b' \rangle \leq G \end{aligned}$$

This shows that $\langle a', b' \rangle \lhd G$ is a normal subgroup. Consider the subgroup $\mathbb{Z}/3\mathbb{Z} \cong \langle a \rangle \leq G$. It follows that

(i) $\langle a', b' \rangle \lhd G$
(ii) $\langle a \rangle \cap \langle a', b' \rangle = 1$;
(iii) $\langle a \rangle \cdot \langle a', b' \rangle = G$ since $b = b'a^2$ so this product of subgroups contain the generators of G;

Hence G is an internal semi direct product of $\langle a', b' \rangle$ with $\langle a \rangle$, and so G is of order $|\langle a', b' \rangle| \cdot |\langle a \rangle| = 4 \cdot 3 = 12$.

(b) Using the first two relations, we observe that elements of G are words that alternates between a or a^2 with b or b^2. Since $(ab)^2 = 1$, we infer that $bab = a^2$ and $aba = b^2$. We count the number of words starting with a of the form $a^{k_1}b^{k_2}a^{k_3}b^{k_4} \dots a^{k_r}$ by length r. There are two different words of length 1. There are at most four different words of length 2: of the form $a^{k_1}b^{k_2}$. By the above relations, strings containing only a's and b's (of power 1) can be reduced to words of length 1 or 2. So strings of length 3 must contain a power of 2 in the middle. Since $b^2a^2 = ab$, strings of length 3 are of the form ab^2a or a^2b^2a. By a similar argument, every string of length 4 can be reduced to a smaller length. Hence there are at most 10 words starting with a. A similar argument shows the same for words starting by b. We conclude that the number of elements in G is bounded by 23, as desired.

(3) A_5 is a simple group of order 60, hence it has no group of order 30. Let $\sigma = (12345)$ and $\tau = (12)(34)$. If we show that $\langle \sigma, \tau \rangle$ has at least 16 elements, it would show that $A_5 = \langle \sigma, \tau \rangle$. We have that

$$\begin{aligned} \tau\sigma\tau^{-1} &= (21435) \\ \sigma\tau &= (135) \\ \tau\sigma &= (245) \\ \sigma^2\tau &= (14523) \end{aligned}$$

The three 5-cycles generates subgroups of order 5 which intersect trivially, hence they generate $4 \cdot 3 + 1 = 13$ distinct elements of A_5. The two 3-cycles generates 4 more distinct elements, and so $\langle \sigma, \tau \rangle$ has at least 17 elements, as desired.

(4) Suppose that F_S is solvable. Choose two distinct generators $a, b \in S$ and let $R = S \backslash \{a, b\}$. We obtain that $F_S/R = \langle S|R \rangle = \langle a, b| \rangle = F_{\{a,b\}}$ is a free group generated by two elements (exercise 4). We observed in the previous point that A_5 can be generated by two elements, say $\alpha, \beta \in A_5$. The universal property of free groups tells us that there is a unique group homomorphism $f : F_{\{a,b\}} \to A_5$ such that $f(a) = \alpha$ and $f(b) = \beta$. Since

those elements generates A_5, we obtain that f is surjective and thus $A_5 \cong F_{\{a,b\}}/\ker(f)$. Since quotients of solvable groups are solvable, this would imply that A_5 is solvable. This is a contradiction since A_5 is simple.