GROUP THEORY 2024 - 25, SOLUTION SHEET 11

Exercise 1. To do yourself. Ask the assistant if something is unclear.

Exercise 2. Let $1 = G_0 \lhd G_1 \lhd \ldots \lhd G_{k-1} \lhd G_k = G$ be a central series for G. Suppose $G_1 \neq 1$, otherwise remove it from the chain until this condition is satisfied. This chain being a central series implies that

$$1 \neq G_1 = G_1/G_0 \leq Z(G/G_0) = Z(G).$$

In particular G has a non trivial center.

Exercise 3. By the assumption that H is normal, we have that $[G, H]$ is a normal subgroup of G. By minimality, it must be either 1 or H itself. If it is 1 then we are done. If not, we can inductively show that $[G, [G, \ldots [G, H] \ldots]] = H$. However, this commutator group is always inside of some G^i (term of a central series of G), hence must eventually be equal to 1, which contradicts the assumption.

Exercise 4. (1) For $n \geq 3$ the center $Z(S_n) = 1$ is trivial, hence S_n can't be nilpotent by the first exercise. S_1 and S_2 are abelian, hence nilpotent.

(2) We start by some preliminaries observations. First recall that the center of the dihedral group is given by

$$Z(D_{2n}) = \begin{cases} 1 & \text{if } n \text{ is odd;} \\ \langle r^{n/2} \rangle \cong \mathbb{Z}/2\mathbb{Z} & \text{if } n = 2k \text{ is even;} \end{cases}$$

where r is the rotation of order n. You have seen this in algebraic structures, you are invited to prove it again if you don't recall it. By the first exercise we know that if the center is trivial, the the group can't be nilpotent. Secondly, when $n = 2k$ is even, we observe that $D_{2\cdot(2k)}/Z(D_{2\cdot(2k)}) \cong D_{2k}$. To prove it, remember that

$$D_{2n} \cong \mathbb{Z}/n\mathbb{Z} \rtimes_{\varphi} \mathbb{Z}/2\mathbb{Z} \tag{1}$$

where the action is given by $b \cdot a = ba$ for all $a \in \mathbb{Z}/n\mathbb{Z}$ and $b \in \{-1, 1\} = \mathbb{Z}/2\mathbb{Z}$. You can easily check by hand that

$$\psi : D_{2\cdot(2k)} = \mathbb{Z}/2k\mathbb{Z} \rtimes_{\varphi} \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/k\mathbb{Z} \rtimes_{\varphi} \mathbb{Z}/2\mathbb{Z} = D_{2k}$$
$$(a, b) \mapsto (a, b)$$

defines a surjective group homomorphism. We notice that the kernel of this map is $\ker(\psi) = \langle (k, 0) \rangle \times_{\varphi} \{1\}$, which corresponds to $\langle r^k \rangle = Z(D_{2n})$ through the isomorphism (1). By the first isomorphism theorem, we get that $D_{2n}/Z(D_{2n}) \cong D_n$ as desired.

We are now ready to prove that D_{2n} is nilpotent if and only if $n = 2^k$ for some k.

($\Longrightarrow$): Suppose that D_{2n} is nilpotent. By the above we know that it has a non trivial center, which implies that $n = 2k_1$ is even. Since the quotient of a nilpotent group is nilpotent, its quotient $D_{2k_1} = D_{2n}/Z(D_{2n})$ is nilpotent as well. We can repeat the same argument to obtain that $k_1 = 2k_2$ must be even as well, so that $n = 2^2k_2$. This process must end after a finite number of steps, which shows that $n = 2^k$ for some integer k.

($\Leftarrow$): We prove by induction that $D_{2\cdot 2^k}$ is nilpotent. For $k = 0$ the result is clear. Suppose that $D_{2\cdot 2^{k-1}}$ is nilpotent. By the above the center is non trivial and we have a short exact sequence

$$1 \to Z(D_{2\cdot 2^k}) \to D_{2\cdot 2^k} \to D_{2\cdot 2^{k-1}} \to 1.$$

We conclude using the induction hypothesis and Theorem 38.

Exercise 5. If G is nilpotent, then by the normalizer property we deduce directly that each maximal subgroup of G is normal in G.

Conversely, let P be a Sylow p-subgroup of G. If P is not normal in G, let M be a maximal subgroup containing its normalizer (observe that we can choose such a subgroup because G is finite). By hypothesis, M must be normal. By exercise 7 of series 9, we have $N_G(P)M = G$, which contradicts the fact that $N_G(P) \subset M$, because then we would have $N_G(P)M \subset M \neq G$. We deduce that all Sylow subgroups of G are normal. By theorem 11 of the lecture notes and exercise 3 of series 9, we conclude that G is nilpotent.

Exercise 6. Let $m_1, \dots, m_r$ be the maximal subgroups of G (finite number, because G is finite). Observe that we have an action of G on the set $\{m_1, \dots, m_r\}$ given by conjugation (check that this is indeed an action). Hence, if $x \in J = \bigcap_{i=1}^r m_i$, then for all $a \in G$, $axa^{-1} \in \bigcap_{i=1}^r am_ia^{-1}$. The second intersection is equal to $\bigcap_{i=1}^r m_i$ by the above observation on the action, so we get $axa^{-1} \in \bigcap_{i=1}^r m_i = J$. By arbitrary choice of $a \in G$, we conclude that J is normal in G.

Now, let P be a Sylow p-subgroup of J. By the same exercise of series 9 as above, $N_G(P)J = G$. If $N_G(P) \neq G$, then $N_G(P)J$ is contained in a maximal subgroup of G, so the above said cannot hold. Hence, $N_G(P) = G$, meaning that P is normal in G and, in particular, normal in J. By the same reasoning as in the above exercise, we conclude that J is nilpotent.