GROUP THEORY 2024 - 25, EXERCISE SHEET 7

Exercise 1. (hard) *To always do in every course!*
Review the lecture and understand/fill in the gaps in the proofs.

Exercise 2. (easy) Show that a normal subgroup $H \subseteq G$ is a maximal normal subgroup of G if and only if G/H is a simple group.

Exercise 3. (easy) Determine a composition series for $\mathbb{Z}/12\mathbb{Z}$. List and identify each composition factor attached to your composition series. Is the composition series unique ? Are the composition factors unique ?

Exercise 4. (medium) Does A_4 have a composition series ? If so, find one and identify the composition factors. Can you deduce a composition series for S_4 ? If so, identify the composition factors.

Exercise 5. (easy) What are the composition factors of a semi-direct product $G \rtimes_\varphi H$ in terms of the composition factors of G and H?

Exercise 6. (medium)

(1) Let n be a positive integer and let $n = p_1^{a_1} \cdot ... \cdot p_k^{a_k}$ be its prime factorisation, then find a composition series for $\mathbb{Z}/n\mathbb{Z}$. With multiplicities, what are the composition factors?
(2) Assuming the Jordan-Hölder theorem, prove that every $n \in \mathbb{N}$ can be uniquely decomposed as a product of powers of prime numbers.

Exercise 7. (medium) Find a composition series for D_{2n} and list the composition factors. What is the length of the composition series?

Exercise 8. (medium) *Constructing an infinite simple group*
Let

$$G_1 \subseteq G_2 \subseteq G_3 \subseteq ...$$

be a sequence of simple groups, then show that

$$G := \bigcup_{i=1}^{\infty} G_i$$

is a simple group. Using this, give an example of an infinite simple group.

Notation: Let G be a group. If G has a composition series $1 = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_n = G$, then we let $\operatorname{length}(G) := n$. This is well defined by a theorem seen in class.

Let G be a group. An ascending normal chain of G is a chain of subgroups of G of the type

$$G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \dots$$

Descending normal chains are defined in the same way, with reverse inclusions.

Exercise 9. (hard) Let G be a group. For $1, 2, 3$ assume that G has finite length.

(1) Consider a short exact sequence

$$1 \to K \to G \to L \to 1$$

Prove that $\operatorname{length}(K), \operatorname{length}(L) < +\infty$ and that $\operatorname{length}(K) + \operatorname{length}(L) = \operatorname{length}(G)$.

(2) Prove that if $K \trianglelefteq G$ is a *proper* normal subgroup, then $\operatorname{length}(K) < \operatorname{length}(G)$.

(3) Prove that all ascending and descending normal chains made up of only normal subgroups of G are finite and the number of different subgroups appearing in such a chain is at most $\operatorname{length}(G) + 1$.

(4) Prove that a group G has finite length if and only if the following two conditions hold:
   - (a) Any descending normal chain starting with a normal subgroup H of G stabilizes, i.e. there exists $n \in \mathbb{N}$ such that $G_i = G_n$ for all $i \geq n$.
   - (b) If H is a normal subgroup of G, then all ascending normal chains of H made up of only normal subgroups of H stabilize.