GROUP THEORY 2024 - 25, EXERCISE SHEET 6

Exercise 1. (hard) *To always do in every course!*
Review the lecture and understand/fill in the gaps in the proofs.

Exercise 2. (easy) Classify all abelian groups of the following orders up to isomorphism:

(1) 4
(2) 6
(3) 180
(4) 72
(5) 200.

Exercise 3. (easy) *Classification of finite abelian groups*

(1) Let A be an abelian group of order 100 that contains no element of order 4. Prove that A has a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
(2) Let p be a prime number. How many abelian groups of order p^5 are there, up to isomorphism? More generally, how many abelian groups of order p^n are there for an arbitrary $n \in \mathbb{N}$?

Given a family of abelian groups $(A_i)_{i\in I}$ we define their direct product as the abelian group

$$\prod_{i\in I} A_i = \{(a_i)_{i\in I} \mid a_i \in A_i, \text{for all } i\}$$

where the addition is performed component wise.

Exercise 4. (easy)
Let $\{A_\alpha | \ \alpha \in I\}$ a set of abelian groups. Show that

(1) $\bigoplus\limits_{\alpha\in I} \mathrm{Tors}(A_\alpha) \cong \mathrm{Tors}(\bigoplus_{\alpha\in I} A_\alpha)$;
(2) $\mathrm{Tors}(\prod_{\alpha\in I} A_\alpha) \subseteq \prod\limits_{\alpha\in I} \mathrm{Tors}(A_\alpha)$;

Find an example which shows that the inclusion of the second point can be strict.

Exercise 5. (medium)
Let $d = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ where $p_1, \dots, p_k$ are distinct primes and $a_1, \dots, a_k$ are positive integers. Show that

$$\mathbb{Z}/d\mathbb{Z} \cong \mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \mathbb{Z}/p_2^{a_2}\mathbb{Z} \times \dots \times \mathbb{Z}/p_k^{a_k}\mathbb{Z}.$$

Hint: Start by showing that if $gcd(a, b) = 1$, then $\mathbb{Z}/ab\mathbb{Z} \cong \mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z}$.

Let A be an abelian group. We say that an element $a \in A$ is n- **divisible** for an integer n if there exists $b \in A$ such that $a = nb$. We say that A is n-**divisible** if all elements of A are n-divisible. Furthermore, we say that A is **divisible** if A is n-divisible for all integers n.

Exercise 6. (medium) *Divisible abelian groups*

(1) Give an example of a divisible abelian group.
(2) Give an example of a finite abelian group which is 2-divisible but not 3-divisible.
(3) Give an example of an infinite abelian group which is 2-divisible but not 3-divisible.
(4) Prove that a finite divisible abelian group must be trivial.

Exercise 7. (hard)
Let G be a finitely generated abelian group that fits in the exact sequence

$$0 \to \mathbb{Z} \to G \to \mathbb{Z}/12\mathbb{Z} \to 0.$$

Classify G up to isomorphism.

Hint: Apply the classification theorem for finitely generated abelian groups and study the short exact sequence.

Exercise 8. (hard)
Let G be a (not necessarily abelian) p-group of order p^n, where p is a prime number and $n \in \mathbb{N}^*$

(1) Prove without using any theorem that makes the statement trivial that G has an element of order p.
(2) Prove that G has (at least) a normal subgroup of order p^k for all $0 \leq k \leq n$.
    Hint: Proceed by induction. If $N \trianglelefteq G$ is of order p^{k-1}, show that $Z(G/N)$ contains an element of order p (something in sheet 4 can help you). Conclude by studying $\pi : G \to G/N$.
(3) Prove that there exists an integer m and a (finite) chain of nested normal subgroups of G

$$\{e\} = G_m \trianglelefteq G_{m-1} \trianglelefteq \ldots \trianglelefteq G_0 = G$$

such that each of the quotients G_{i-1}/G_i is abelian for $1 \leq i \leq n$.

Hint: Use induction and the previous point.
Note: a group that admits such a chain is called **solvable**. You do not need any of the results for solvable groups (which you will see later on) to prove the statement.