GROUP THEORY 2024 - 25, EXERCISE SHEET 5

Exercise 1. *To always do in every course!*
Review the lecture and understand/fill in the gaps in the proofs.

Exercise 2. *Computing some examples of torsion subgroups*
Determine $\mathrm{Tors}(A)$ for the following examples of abelian groups:

(1) A is a finite abelian group.
(2) $A = (\mathbb{Q}, +)$.
(3) $A = (\mathbb{Q}/\mathbb{Z})$.
(4) $A = \mathbb{C}^\times$.
(5) A is a subgroup of $\mathbb{Z}$.
(6) A is a subgroup of $\mathbb{Z}^k$ for $k \geq 2$.

Exercise 3. Show that if G is abelian and finitely generated such that $\mathrm{Tors}(G) = G$, then G is a finite group.

Exercise 4. *Free abelian groups*

Given a family of abelian groups $(A_i)_{i\in I}$ we define their direct sum as the abelian group

$$\bigoplus_{i\in I} A_i = \{(a_i)_{i\in I} | a_i \in A_i, \text{for all } i, \text{only finitely many of the } a_i \text{ are non-zero}\}$$

where the addition is performed component wise.

Prove that the following affirmations are equivalent for an abelian group A:

(1) A is a free abelian group. That is

$$A \cong \mathbb{Z}^{\oplus I} := \bigoplus_{i\in I} \mathbb{Z}.$$

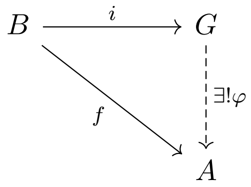
For some indexing set I (not necessarily finite).

(2) There exists a set I and a subset $B = \{a_i | i \in I\} \subset A$ called a **basis**, such that all elements $x \in A$ can be uniquely written as finite sums

$$x = \sum_{k\in I} n_k a_k$$

where all but finitely many n_k equal 0. Note that the uniqueness condition implies that the elements of a basis are linearly independent over $\mathbb{Z}$.

Exercise 5. *Universal property of free abelian groups*
Show that an abelian group G is free if and only if there exists a subset $B \subset G$ satisfying the follow universal property: for all abelian groups A and all set function $f : B \to A$, there exists a unique homomorphism of groups $\varphi : G \to A$ such that $\varphi \circ i = f$:



where $i : B \to G$ is the set inclusion. Note that the subset B is a **basis** of the free abelian group G (see previous exercise).

Exercise 6. Let $F = \mathbb{Z}^3$ and define a function $f : F \to \mathbb{Z}^2$ on a basis (e_1, e_2, e_3) by

$$f(e_1) = (1, 0);\ f(e_2) = (1, 1),\ f(e_3) = (0, -1).$$

Using the universal property of free abelian group, show that you can extend uniquely f to a group homomorphism. Is the image of f free abelian ?

Exercise 7. Recall that the rank of a finitely generated free abelian group A is the positive integer r such that

$$A \cong \mathbb{Z}^r.$$

Also recall that it was shown in class that if $A \subseteq \mathbb{Z}^r$ is a subgroup, then $A \cong \mathbb{Z}^k$ for some $k \leq r$. We will also see in the next exercise that the rank of a free-abelian group is well defined.

Compute the rank of the following free abelian groups:

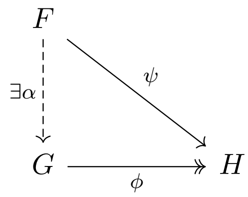
(1) Subgroup generated by $(1, 1)$ in $\mathbb{Z}^2$.
(2) Subgroup generated by $(1, 2)$ and $(-3, -6)$ in $\mathbb{Z}^2$.
(3) $\{a + b\sqrt{2} + c\sqrt{3} \mid a, b, c \in \mathbb{Z}\}$ as an additive subgroup of $\mathbb{R}$.
(4) Subgroup generated by $(2, 3, 8), (1, 5, 1)$ and $(1, -9, 34)$ in $\mathbb{Z}^3$.
(5) Subgroup generated by $(2, 3, 8), (1, 5, 1)$ and $(1, -9, 13)$ in $\mathbb{Z}^3$.

Exercise 8. Show that if $\mathbb{Z}^n \cong \mathbb{Z}^m$, then $n = m$.
Hint: You know from linear algebra that this statement is true if instead of $\mathbb{Z}$ we had a field k and a k - linear isomorphism. Can you reduce to this case by quotienting by appropriate subgroups?

Exercise 9. Show that the positive rationals $\mathbb{Q}^{>0}$ with group law given by usual multiplication is not a finitely generated abelian group. However show that it is a free-abelian group by exhibiting a basis.

For the ones of you more interested in algebra:

Exercise 10. Prove that a finitely generated abelian group F is free if and only if for all pairs (ϕ, ψ), where $\phi : G \to H$ is a surjective homomorphism between two abelian groups G and H, and $\psi : F \to H$ is a homomorphism, there exists a homomorphism $\alpha : F \to G$ such that $\psi = \phi \circ \alpha$:



Note: We call an abelian group which satisfies the above property projective. This exercise proves that being projective is the same as being free in the case of finitely generated abelian groups.

Exercise 11. Note that a homomorphism $A \to B$ of abelian groups induces a map

$$\text{Tors}(A) \to \text{Tors}(B).$$

Now let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of abelian groups. Determine if

$$0 \to \text{Tors}(A) \to \text{Tors}(B) \to \text{Tors}(C) \to 0$$

is also exact in general.