GROUP THEORY 2024 - 25, EXERCISE SHEET 3

The first 3 exercises of this sheet are important results from *Structures algébriques*. If you feel uncomfortable with these results, we encourage you to take the time to prove them since ideas from their proofs will recur many times throughout the course. Feel free to skip these exercises if you feel you have some level of confidence in applying these results.

Exercise 1. *Normal Subgroups and Group Quotients*
Let G be a group and $H \subseteq G$ be a subgroup. Show that the following statements are equivalent:

(1) The subgroup H is a normal subgroup of G.
(2) Whenever $a_1, a_2, b_1, b_2 \in G$ are such that the left co-sets $a_1H = b_1H$ and $a_2H = b_2H$, then the coset $a_1a_2H = b_1b_2H$. In other words, the assignment

$$\begin{aligned} \cdot : G/H \times G/H &\to G/H \\ (g_1H, g_2H) &\mapsto g_1g_2H. \end{aligned}$$

is a well-defined function.

Furthermore, using the above statement, show that if H is a normal subgroup of G then the set of left co-sets G/H has a natural group structure.

Exercise 2. *First Isomorphism Theorem*
Let $\phi : G \to F$ a homomorphism and let $H \trianglelefteq G$ be a normal subgroup of G such that $H \subseteq ker\phi$.

(1) *Universal property of the quotient group*
There exists a unique homomorphism $\bar{\phi} : G/H \to F$ such that the following diagram commutes (which means that $\bar{\phi} \circ q = \phi$):

$$\begin{array}{ccc} G & \xrightarrow{q} & G/H \\ & \searrow^{\phi} & \downarrow \bar{\phi} \\ & & F \end{array}$$

where $q : G \to G/H$ is the quotient homomorphism

(2) *First isomorphism theorem*
If $H = ker\phi$, then $\bar{\phi}$ is injective and induces an isomorphism $G/H \xrightarrow{\cong} im\phi$.

Exercise 3. *Correspondence Theorem and the Third Isomorphism Theorem*
Let $H \trianglelefteq G$ be a normal subgroup and let $q : G \to G/H$ be the quotient homomorphism.

(1) Show that there is a correspondence between subgroups of G that contains H and subgroups of the quotient G/H. That is show that the following assignments form inverse functions between those two sets:

$$\{\text{subgroups } F \leq G \mid H \leqslant F \leqslant G\} \longleftrightarrow \{\text{subgroups } K \leqslant G/H\}$$

$$F \longmapsto q(F) = F/H$$

$$q^{-1}(K) \longleftarrow K$$

(2) Show that the above correspondence restricts to a correspondence between normal subgroups. That is F is a normal subgroup of G containing H if and only if $q(F)$ is a normal subgroup of G/H.
(3) Suppose that F is a normal subgroup of G containing H then show that

$$G/F \cong \frac{G/H}{q(F)} = \frac{G/H}{F/H}$$

Exercise 4. *Second isomorphism theorem*
Let G be a group and $H, F \leqslant G$ be subgroups such that $F \leqslant N_G(H)$. Show that:

(1) $F \cap H \trianglelefteq F$ and $H \trianglelefteq FH$
(2) We have an isomorphism $F/F \cap H \xrightarrow{\cong} FH/H$.

Note: Recall that the *Normalizer of* H *in* G is defined as $N_G(H) = \{g \in G : gH = Hg\}$ and is a subgroup of G.

Recall that a group action is defined to be a group homomorphism of the form $G \to \mathrm{Bij}(X)$. In the following exercise we propose another (equivalent) definition of a group action.

Exercise 5. *Equivalence of definitions of group actions: Very important! To remember and use in practice!*
Show that an action is precisely the data of a map $\cdot : G \times X \to X$ such that

(1) $e_G \cdot x = x$ for all $x \in X$;
(2) $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G$ and $x \in X$.

More precisely, you need to show that there is a bijection of sets

$$\{\Phi : G \to \mathrm{Bij}(X) \mid \Phi \text{ is an action}\} \cong \{\cdot : G \times X \to X \mid (1) \ \&\ (2) \text{ hold}\}.$$

Then practice going from one representation of a group action to the other by computing it explicitly for the different actions you have seen in the last exercise sheets until you feel comfortable.

Exercise 6. Let $\{e_G\} \neq H \leq G$ be a subgroup and recall the action $G \times G/H \to G/H$ from the last exercise's sheet. Show that if H is normal in G, then the action on G/H is *not* faithful.

Exercise 7. Let G be a group and let H be a subgroup of G with index 2. Recall that the index of a subgroup H is the cardinality of the set G/H. Prove that this implies that H is a normal subgroup of G.

Exercise 8. *Some properties of cosets useful in practice*
Let $H, K \leq G$ be subgroups of a group G. Show that

(1) $gH = g'H$ if and only if $g'^{-1}g \in H$;
(2) $gH \cap g'H \neq \emptyset$ implies that $gH = g'H$;
(3) $gH \cap g'K$ is either empty or a coset of $H \cap K$.

For the following exercise, recall that two G-sets X and Y are isomorphic as G-sets if there exists a bijection $f : X \to Y$ such that $f(g \cdot x) = g \cdot f(y)$.

Exercise 9. *A bit harder...*
Let G be a group. Let $\mathcal{X}$ be the set of transitive G-actions.

(1) Show that if X is isomorphic to Y as G sets, then $G \curvearrowright X$ is transitive if and only if $G \curvearrowright Y$ is transitive.
(2) Deduce that isomorphism of G-sets defines an equivalence relation $\sim$ on $\mathcal{X}$.
(3) Show that there is a bijection between the set of conjugacy classes of subgroups of G and isomorphism classes of transitive G-actions. That is show that there is a bijection

$$\{ \text{ Conjugacy class of subgroups } H \leq G \text{ } \} \cong \mathcal{X}/\sim$$

Use this to describe the classes of transitive G-actions for the groups $G = \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/8\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, S_3$.