GROUP THEORY 2024 - 25, EXERCISE SHEET 12

Exercise 1. (hard) *To always do in every course!*
Review the lecture and understand/fill in the gaps in the proofs.

Exercise 2. (easy) Let

$$1 \to H \to G \to F \to 1$$

be a short exact sequence of groups such that F is a free group. Show that the sequence splits.

Exercise 3. (easy) Let F be a free group on a set S, where $|S| \geq 2$. Show that F is torsion free and has trivial centre.

Exercise 4. (easy) Let X and Y be disjoint sets. Let N be the normal subgroup of $F_{X \cup Y}$ generated by Y. Show that $F_{X \cup Y}/N \cong F_X$.

Exercise 5. *The Quaternion Group* (medium)
Consider the group:

$$Q_8 := \left\langle i, j, k \;| i^4 = 1,\, i^2 = j^2 = k^2 = ijk \right\rangle .$$

The group Q_8 is called the Quaternion group.

(1) Define the element $-1 := i^2$. Show that -1 is in the the centre of Q_8.

(2) Show that Q_8 is a finite group of order 8.

(3) Find the centre of Q_8.

(4) Show that all subgroups of Q_8 are normal subgroups.

Exercise 6. (medium)
Compute the abelianisations of the following groups:

(1) A_5

(2) A_4

(3) S_n for $n \geq 5$.

(4) F_S where $|S| = 2$.

(5) $\langle a, b \mid a^2b^3,\, a^4b^5 \rangle$.

Exercise 7. (Hard)

(1) Show that $\langle a, b \mid a^2 = 1,\, b^2 = 1,\, (ab)^2 = 1 \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

(2) Show that $\langle a, b \mid a^3 = 1,\, b^3 = 1,\, (ab)^2 = 1 \rangle \cong A_4$.

(3) Prove that A_5 can be generated by an element of order 5 and an element of order 2.

(4) Prove that F_S is not solvable whenever $|S| \geq 2$.