GROUP THEORY 2024 - 25, EXERCISE SHEET 10

Exercise 1. (hard) *To always do in every course!*
Review the lecture and understand/fill in the gaps in the proofs.

Exercise 2. (Medium) *Subgroups of prime index*
Let G be a finite group with subgroup $H \leq G$ of prime index p, and suppose that p is the smallest prime factor of $|G|$. Show that H is normal in G.
Hint: Consider $G \curvearrowright G/H$ and show that $\ker(G \to S_p) = H$ by showing that those two subgroups have the same index.

Exercise 3. (Medium)

(1) Let p be a prime number. Show that a group of order p^n is simple if and only if $n = 1$.

(2) Let p, q be distinct primes. Show that a group of order pq cannot be simple.
Hint: Consider the two cases $p > q$ and $q > p$.

(3) Let p, q be distinct primes, show that there is no simple group of order p^2q.

(4) Let p, q, r be distinct primes, show that there is no simple group of order pqr.

Exercise 4. (medium) Consider the following celebrated result known as Burnside's Theorem:

Theorem: Let p, q be prime numbers, then any group of order $p^a q^b$ is solvable.

Assuming Burnside's Theorem prove that non-abelian groups of order less than 60 cannot be simple.

Exercise 5. (Medium) Let $n \geq 2$. Prove that a group of order $2^n \cdot 3$ has a non-trivial proper normal subgroup.

Exercise 6. (Hard)

(1) Let L be a finite cyclic group, K a group, $\varphi_1, \varphi_2 : L \to \mathrm{Aut}(K)$ two group actions, and suppose that φ_2 or φ_1 is injective. Show that if $\varphi_1(L)$ and $\varphi_2(L)$ are conjugate subgroups of $\mathrm{Aut}(K)$, then the two induced semi direct products are isomorphic:

$$K \rtimes_{\varphi_1} L \cong K \rtimes_{\varphi_2} L$$

Hint: Let $\sigma \in \mathrm{Aut}(K)$ such that $\sigma\varphi_1(L)\sigma^{-1} = \varphi_2(L)$. You can find $a \in \mathbb{N}$ such that $\sigma \circ \varphi_1(l) \circ \sigma^{-1} = \varphi_2(l)^a$ for all $l \in L$. Define $K \rtimes_{\varphi_1} L \to K \rtimes_{\varphi_2} L$ using σ and a.

Remark. The injectivity hypothesis is not required for the statement to hold, however we'll assume it since it holds for our two applications.

(2) Show that there exists a unique (up to isomorphism) non trivial semi direct product of the form $\mathbb{Z}/p^2\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}$.
  Hint: The group $\mathrm{Aut}(\mathbb{Z}/p^2\mathbb{Z})$ is cyclic (you don't need to prove it).

(3) Show that there exists a unique (up to isomorphism) non trivial semi direct product of the form $(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}) \rtimes \mathbb{Z}/p\mathbb{Z}$
  Hint: Show that every automorphism of K is L-linear, and then use Sylow's theorems.

Exercise 7. *(Hard) Classification of groups of order p^3*
Let p be an odd prime. We have seen that every group of order p^2 is either isomorphic to $\mathbb{Z}/p^2\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. In this exercise we want to classify groups of order p^3.

(1) Classify all abelian groups of order p^3.

Suppose now that G is not abelian.

(2) Suppose that G has an element of order p^2. Show that

$$G \cong \mathbb{Z}/p^2\mathbb{Z} \rtimes_\varphi \mathbb{Z}/p\mathbb{Z}$$

for some $\varphi : \mathbb{Z}/p\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/p^2\mathbb{Z})$. Conclude that there is a unique non abelian group of order p^3 containing an element of order p^2.

(3) Suppose now that G does not have an element of order p^2. Show that

$$G \cong (\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}) \rtimes_\varphi \mathbb{Z}/p\mathbb{Z}$$

for some $\varphi : \mathbb{Z}/p \to \mathrm{Aut}(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z})$. Conclude that there exists a unique non abelian group of order p^3 not containing an element of order p^2. Note that in that case every non-trivial element is of order p.

(4) Classify all groups of order p^3.

For those interested, here is a construction of the non-abelian groups of order p^3:

(5) Let G be the the group

$$G = \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} | a, b \in \mathbb{Z}/p^2\mathbb{Z} \text{ and } a \equiv 1 \pmod p\}$$

endowed with matrix multiplication. Show that G is a non abelian group of order p^3 isomorphic to the unique non trivial semi direct product $\mathbb{Z}/p^2\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}$.

(6) Let G be the Heisenberg group

$$\mathrm{Heis}(\mathbb{Z}/p\mathbb{Z}) = \{\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} | a, b, c \in \mathbb{Z}/p\mathbb{Z}\}$$

endowed with matrix multiplication. Show that G is a non-abelian group of order p^3 isomorphic to the unique non trivial semi direct product $(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}) \rtimes \mathbb{Z}/p\mathbb{Z}$.