MATH-207(d) Analysis IV
Exercise session 9

Exercice 1. Let $\alpha \in \mathbb{R}$. Use the residue theorem to compute the following integral

$$\int_{-\infty}^{\infty} \frac{\sin(\alpha x)}{1+x^2} \mathrm{d}x. \tag{1}$$

You are *not* allowed to use the fact that the sine function is odd.

Answer. This exercise is very similar to Exercise 2 from the previous Exercise sheet with an additional complication due to the fact that $\alpha \in \mathbb{R}$ rather than $\alpha > 0$. As discussed in the solution to that problem, one strategy to tackle this integral is to first recognise that we can write

$$\int_{-\infty}^{\infty} \frac{\sin(\alpha x)}{1+x^2} \mathrm{d}x = \int_{-\infty}^{\infty} \frac{\mathrm{Im}(e^{i\alpha x})}{1+x^2} \mathrm{d}x = \mathrm{Im}\Big(\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{1+x^2} \mathrm{d}x \Big). \tag{2}$$

Let us introduce the function

$$f_\alpha(z) = \frac{e^{i\alpha z}}{1+z^2},$$

It is readily seen that f_α is holomorphic on $\mathbb{C} \setminus \{+i, -i\}$ and has simple poles at $z = +i$ and $z = -i$ with residues

$$\mathrm{Res}_{+i}(f_\alpha) = \frac{e^{-\alpha}}{2i} \qquad \mathrm{Res}_{-i}(f_\alpha) = \frac{e^{\alpha}}{2i}$$

We now consider two seperate cases depending on the value of α.

$\boxed{\alpha \geq 0}$

For this case, as seen in the lecture, we have

$$\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{1+x^2} \mathrm{d}x = 2\pi i \left(\mathrm{Res}_{+i}(f_\alpha)\right) = 2\pi i \left(\frac{e^{-\alpha}}{2i} \right) = \pi e^{-\alpha}. \tag{3}$$

Consequently, if $\alpha \geq 0$ we have that

$$\int_{-\infty}^{\infty} \frac{\sin(\alpha x)}{1+x^2} \mathrm{d}x = \mathrm{Im}\Big(\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{1+x^2} \mathrm{d}x \Big) = \mathrm{Im}\big(\pi e^{-\alpha}\big) = 0.$$

$\boxed{\alpha < 0}$

Since $\alpha < 0$ by assumption, it might appear at first glance that we cannot immediately apply the results of the lecture in this case. However, a nice trick allows us to circumvent this difficulty.

Notice that if $\alpha < 0$, we can write

$$\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{1+x^2}\mathrm{d}x = \int_{-\infty}^{\infty} \overline{\frac{e^{-i\alpha x}}{1+x^2}}\mathrm{d}x = \overline{\int_{-\infty}^{\infty} \frac{e^{-i\alpha x}}{1+x^2}\mathrm{d}x} = \overline{\int_{-\infty}^{\infty} \frac{e^{i\beta x}}{1+x^2}\mathrm{d}x}, \tag{4}$$

where we have introduced the constant $\beta > 0$ as $\beta = -\alpha$, and for any $z \in \mathbb{C}$, we denote by $\overline{z}$ the complex conjugate of z.

Since $\beta > 0$, we can now apply the results from the lecture to deduce once again that

$$\int_{-\infty}^{\infty} \frac{e^{i\beta x}}{1+x^2}\mathrm{d}x = 2\pi i\left(\mathrm{Res}_{+i}(f_\beta)\right) = 2\pi i\left(\frac{e^{-\beta}}{2i}\right) = \pi e^{-\beta}. \tag{5}$$

Consequently, if $\alpha = -\beta < 0$ we have that

$$\int_{-\infty}^{\infty} \frac{\sin(\alpha x)}{1+x^2}\mathrm{d}x = \mathrm{Im}\Big(\overline{\int_{-\infty}^{\infty} \frac{e^{-i\alpha x}}{1+x^2}\mathrm{d}x}\Big) = \mathrm{Im}\big(\overline{\pi e^{\alpha}}\big) = \mathrm{Im}\big(\pi e^{\alpha}\big) = 0.$$

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Exercice 2. Use the residue theorem to compute the Fourier transform $\hat{f}(\alpha)$ of the function

$$f(x) = \frac{x}{1+x^4}, \tag{6}$$

for all $\alpha \neq 0$.

Hint: The following fact might be useful:

$$z^2 = i \iff z = \pm\frac{1+i}{\sqrt{2}}, \qquad z^2 = -i \iff z = \pm\frac{1-i}{\sqrt{2}}.$$

Answer. Note that by the definition of the Fourier transform (given in Chapter 15 of the course textbook), it holds that

$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{xe^{-i\alpha x}}{1+x^4}\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\overline{\int_{-\infty}^{\infty} \frac{xe^{i\alpha x}}{1+x^4}\mathrm{d}x}, \tag{7}$$

where, for any $z \in \mathbb{C}$, we denote by $\overline{z}$ the complex conjugate of z.

Let us therefore introduce the complex-valued function g as

$$g(z) = \frac{ze^{i\alpha z}}{1+z^4} \qquad \text{for all } z \in \mathbb{C} \quad \text{s.t.} \quad 1+z^4 \neq 0.$$

Using the hint provided in the exercise, we deduce that

$$1+z^4 = (z^2-i)(z^2+i) = \frac{1}{2}\left(z - \frac{1+i}{\sqrt{2}}\right)\left(z - \frac{1-i}{\sqrt{2}}\right)\left(z + \frac{1+i}{\sqrt{2}}\right)\left(z + \frac{1-i}{\sqrt{2}}\right).$$

Consequently, g is holomorphic on $\mathbb{C} \setminus S$ where the set S is defined as

$$S = \left\{\frac{1+i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}, -\frac{1+i}{\sqrt{2}}, -\frac{1-i}{\sqrt{2}}\right\}.$$

Moreover, at each of the four points in S, the function g has a simple pole, and the associated residues are given by

$$\begin{aligned}
\mathrm{Res}_{\frac{1+i}{\sqrt{2}}}(g) &= -\frac{i}{4}e^{-(1-i)\alpha/\sqrt{2}} = \frac{1}{4}\left(-ie^{-\alpha/\sqrt{2}}\cos(\alpha/\sqrt{2}) + e^{-\alpha/\sqrt{2}}\sin(\alpha/\sqrt{2})\right), \\
\mathrm{Res}_{\frac{1-i}{\sqrt{2}}}(g) &= \frac{i}{4}e^{(1+i)\alpha/\sqrt{2}} = \frac{1}{4}\left(ie^{\alpha/\sqrt{2}}\cos(\alpha/\sqrt{2}) - e^{\alpha/\sqrt{2}}\sin(\alpha/\sqrt{2})\right), \\
\mathrm{Res}_{-\frac{1+i}{\sqrt{2}}}(g) &= -\frac{i}{4}e^{(1-i)\alpha/\sqrt{2}} = \frac{1}{4}\left(-ie^{\alpha/\sqrt{2}}\cos(\alpha/\sqrt{2}) - e^{\alpha/\sqrt{2}}\sin(\alpha/\sqrt{2})\right), \\
\mathrm{Res}_{-\frac{1-i}{\sqrt{2}}}(g) &= \frac{i}{4}e^{-(1+i)\alpha/\sqrt{2}} = \frac{1}{4}\left(ie^{-\alpha/\sqrt{2}}\cos(\alpha/\sqrt{2}) + e^{-\alpha/\sqrt{2}}\sin(\alpha/\sqrt{2})\right).
\end{aligned}$$

As in the previous example, we now consider two seperate cases depending on the value of α.

$\boxed{\alpha > 0}$

For this case, as seen in the lecture, we have

$$\int_{-\infty}^{\infty} \frac{xe^{i\alpha x}}{1+x^4}\mathrm{d}x = 2\pi i\left(\mathrm{Res}_{\frac{1+i}{\sqrt{2}}}(g) + \mathrm{Res}_{-\frac{1-i}{\sqrt{2}}}(g)\right) = 2\pi i\left(\frac{1}{2}e^{-\alpha/\sqrt{2}}\sin(\alpha/\sqrt{2})\right) \tag{8}$$

$$= \pi i e^{-\alpha/\sqrt{2}}\sin(\alpha/\sqrt{2}). \tag{9}$$

Consequently, if $\alpha > 0$ the Fourier transform $\hat{f}(\alpha)$ is given by

$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{xe^{-i\alpha x}}{1+x^4}\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\overline{\int_{-\infty}^{\infty}\frac{xe^{i\alpha x}}{1+x^4}\mathrm{d}x} = -\sqrt{\frac{\pi}{2}}ie^{-\alpha/\sqrt{2}}\sin(\alpha/\sqrt{2}).$$

$\boxed{\alpha < 0}$

The case $\alpha < 0$ can be dealt with similarly to the previous exercise. Indeed, we note that for $\alpha < 0$ it holds that

$$\int_{-\infty}^{\infty}\frac{xe^{-i\alpha x}}{1+x^4}\mathrm{d}x = \int_{-\infty}^{\infty}\frac{xe^{i\beta x}}{1+x^4}\mathrm{d}x, \tag{10}$$

where we have introduced the constant $\beta > 0$ as $\beta = -\alpha$, and for any $z \in \mathbb{C}$, we denote by $\overline{z}$ the complex conjugate of z.

Since $\beta > 0$, we can now apply the results from the case $\alpha > 0$ computed above to deduce

$$\int_{-\infty}^{\infty}\frac{xe^{i\beta x}}{1+x^4}\mathrm{d}x = \pi i e^{-\beta/\sqrt{2}}\sin(\beta/\sqrt{2}). \tag{11}$$

Consequently, if $\alpha = -\beta < 0$ the Fourier transform $\hat{f}(\alpha)$ is given by

$$\begin{aligned}
\hat{f}(\alpha) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{xe^{-i\alpha x}}{1+x^4}\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{xe^{i\beta x}}{1+x^4}\mathrm{d}x = \sqrt{\frac{\pi}{2}}ie^{-\beta/\sqrt{2}}\sin(\beta/\sqrt{2}) \\
&= -\sqrt{\frac{\pi}{2}}\pi ie^{\alpha/\sqrt{2}}\sin(\alpha/\sqrt{2}).
\end{aligned}$$

Combining the results for the two cases $\alpha > 0$ and $\alpha < 0$ and using the fact that the sine function is odd, we finally have for all $\alpha \neq 0$ that

$$\hat{f}(\alpha) = -\sqrt{\frac{\pi}{2}}ie^{-|\alpha|/\sqrt{2}}\sin(\alpha/\sqrt{2}).$$

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Exercice 3. Let γ be a simple, closed, differentiable curve contained in the disk of radius 2 and centered at $z = 0$ in the complex plane. Use the residue theorem to calculate the following integral.

$$\int_\gamma \tan(z)\ \mathrm{d}z. \tag{12}$$

Answer. We know that the cosine function is zero precisely at complex numbers of the form $z = n\pi + \pi/2$ for any $n \in \mathbb{Z}$. Consequently, the only singularities of $\tan(z)$ inside the disk of radius 2 and centered at $z = 0$ are at the points $z = \pm\pi/2$. Moreover, these singularities are clearly poles of order one and the associated residues at these points are given by

$$\begin{aligned}\mathrm{Res}_{\pi/2}\big(\tan(z)\big) = \lim_{z\to\pi/2}(z-\pi/2)\tan(z) = \lim_{z\to 0} z\tan(z+\pi/2) &= \lim_{z\to 0}\frac{z\sin(z+\pi/2)}{\cos(z+\pi/2)}\\ &= \lim_{z\to 0}\frac{z\cos(z)}{-\sin(z)}\\ &= -\lim_{z\to 0}\frac{z}{\sin(z)} = -1,\end{aligned}$$

and similarly

$$\begin{aligned}\mathrm{Res}_{-\pi/2}\big(\tan(z)\big) = \lim_{z\to-\pi/2}(z+\pi/2)\tan(z) = \lim_{z\to 0} z\tan(z-\pi/2) &= \lim_{z\to 0}\frac{z\sin(z-pi/2)}{\cos(z-\pi/2)}\\ &= \lim_{z\to 0}\frac{-z\cos(z)}{\sin(z)}\\ &= -\lim_{z\to 0}\frac{z}{\sin(z)} = -1.\end{aligned}$$

We now have five cases depending on the nature of the curve γ.

Case 1: $\pm\pi/2 \in \mathrm{int}\gamma$**.**

In this case, the residue theorem yields

$$\int_\gamma \tan(z)\ \mathrm{d}z = 2\pi i\left(\mathrm{Res}_{\pi/2}\big(\tan(z)\big) + \mathrm{Res}_{-\pi/2}\big(\tan(z)\big)\right) = 2\pi i(-2) = -4\pi i.$$

Case 2: $\pi/2 \in \mathrm{int}\gamma$ **and** $-\pi/2 \notin \overline{\mathrm{int}\gamma}$ **.**

In this case, the residue theorem yields

$$\int_\gamma \tan(z)\ \mathrm{d}z = 2\pi i\left(\mathrm{Res}_{\pi/2}\big(\tan(z)\big)\right) = 2\pi i(-1) = -2\pi i.$$

Case 3: $-\pi/2 \in \mathrm{int}\gamma$ **and** $\pi/2 \notin \overline{\mathrm{int}\gamma}$ **.**

In this case, the residue theorem yields

$$\int_\gamma \tan(z)\ \mathrm{d}z = 2\pi i\left(\mathrm{Res}_{-\pi/2}\big(\tan(z)\big)\right) = 2\pi i(-1) = -2\pi i.$$

Case 4: $\pm\pi/2 \notin \overline{\mathrm{int}\gamma}$**.**

In this case, Cauchy's theorem yields that

$$\int_\gamma \tan(z)\ \mathrm{d}z = 0.$$

Case 5: Either $\pi/2 \in \gamma$ or $-\pi/2 \in \gamma$.

In this case, the the integral is ill-defined.

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Exercice 4. Compute the following integral

$$\int_0^{2\pi} \frac{\cos(\theta)\sin(2\theta)}{5+3\cos(2\theta)}\, \mathrm{d}\theta. \tag{13}$$

Hint: A similar exercise was posed in the previous exercise sheet. As before, try to use the residue theorem by recasting this integral as a contour integral on the unit circle. The starting point is to observe that for $z = e^{i\theta}$ we have

$$\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right) \qquad \sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{1}{2i}\left(z-\frac{1}{z}\right).$$

Answer. Taking into account the hints and replicating the discussion in the lecture, we deduce that

$$\int_0^{2\pi} \frac{\cos(\theta)\sin(2\theta)}{5+3\cos(2\theta)}\, \mathrm{d}\theta = \int_0^{2\pi} \frac{1}{4i} \frac{\left(e^{i\theta}+e^{-i\theta}\right)\left(e^{2i\theta}-e^{-2i\theta}\right)}{5+3\left(e^{2i\theta}+e^{-2i\theta}\right)/2}\, \mathrm{d}\theta \tag{14}$$

$$= \int_0^{2\pi} \underbrace{\frac{-1}{2e^{i\theta}} \frac{\left(e^{i\theta}+e^{-i\theta}\right)\left(e^{2i\theta}-e^{-2i\theta}\right)}{10+3\left(e^{2i\theta}+e^{-i2\theta}\right)}}_{:=f(e^{i\theta})} ie^{i\theta}\mathrm{d}\theta \tag{15}$$

$$= \int_\gamma f(z)\, \mathrm{d}z, \tag{16}$$

where γ is the unit circle parameterised by $\theta \mapsto e^{i\theta}$ for $\theta \in [0, 2\pi)$. Consequently, it suffices to study the function

$$f(z) = \frac{-1}{2z}\frac{(z+1/z)\left(z^2-1/z^2\right)}{10+3\left(z^2+1/z^2\right)} = -\frac{1}{2z^2}\frac{(z^2+1)(z^4-1)}{10z^2+3(z^4+1)}.$$

Consider now the polynomial $p(z) = 10z^2 + 3(z^4+1)$. A direct calculation shows that we can factorise this polynomial as

$$10z^2 + 3(z^4+1) = (3z^2+1)(z^2+3).$$

Consequently, the only singularities of the function f inside the unit circle are located at $z = 0, z = i/\sqrt{3}$ and $z = -i/\sqrt{3}$. Moreover, the singularity at $z = 0$ is a pole of order two while the other two singularities are poles of order one. Computing the residues at each of these singularities yields

$$\mathrm{Res}_0(f) = -\frac{1}{2}\lim_{z\to 0}\frac{d}{dz}\left(\frac{(z^2+1)(z^4-1)}{10z^2+3(z^4+1)}\right) = 0,$$

$$\mathrm{Res}_{i/\sqrt{3}}(f) = -\frac{1}{2}\lim_{z\to i/\sqrt{3}}\left(\frac{(z^2+1)(z^4-1)}{3z^2(z+i/\sqrt{3})(z^2+3)}\right) = \frac{i}{6\sqrt{3}}$$

$$\mathrm{Res}_{-i\sqrt{3}}(f) = -\frac{1}{2}\lim_{z\to -i/\sqrt{3}}\left(\frac{(z^2+1)(z^4-1)}{3z^2(z-i/\sqrt{3})(z^2+3)}\right) = -\frac{i}{6\sqrt{3}}.$$

We thus deduce from the residue theorem that

$$\int_0^{2\pi} \frac{\cos(\theta)\sin(2\theta)}{5+3\cos(2\theta)}\,\mathrm{d}\theta = \int_\gamma f(z)\,\mathrm{d}z = 2\pi i \left(\mathrm{Res}_0(f) + \mathrm{Res}_{i/\sqrt{3}}(f) + \mathrm{Res}_{-i\sqrt{3}}(f)\right) = 0.$$

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Exercice 5. Calculate

$$\int_0^{2\pi} \frac{\sin^2(5\theta/2)}{\sin^2(\theta/2)}\,\mathrm{d}\theta. \tag{17}$$

Answer. The computation works analogously to the example in the lecture. However, the denominator vanishes near the boundaries of the interval, and hence the question arises whether the integral is well-defined. To do that, we show that the integrand

$$h(\theta) := \left(\frac{\sin(5\theta/2)}{\sin(\theta/2)}\right)^2 \qquad \forall \theta \in [0, 2\pi)$$

is continuous and bounded over the interval $[0, 2\pi]$. To do so, first note that the sine function is smooth on $\mathbb{R}$ and the function $\sin(\theta/2)$ is zero if and only if $\theta = 2n\pi$ for some integer $n \in \mathbb{Z}$. Hence $\sin(\theta/2) \neq 0$ on the open interval $(0, 2\pi)$, and therefore the function h is continuous on the open interval $(0, 2\pi)$. Next, we want to show that h is also bounded on the closed interval $[0, 2\pi]$. To do so, it suffices to prove that the one-sided end point limits

$$\lim_{\theta\to 0^+} \frac{\sin(5\theta/2)}{\sin(\theta/2)} \qquad \text{and} \qquad \lim_{\theta\to(2\pi)^-} \frac{\sin(5\theta/2)}{\sin(\theta/2)}$$

both exist. There are several possibilities. For example, we can use the theorem of L'Hopital:

$$\lim_{\theta\to 0^+} \frac{\sin(5\theta/2)}{\sin(\theta/2)} = \lim_{\theta\to 0^+} 5\frac{\cos(5\theta/2)}{\cos(\theta/2)} = 5, \tag{18}$$

$$\lim_{\theta\to(2\pi)^-} \frac{\sin(5\theta/2)}{\sin(\theta/2)} = \lim_{\theta\to(2\pi)^-} 5\frac{\cos(5\theta/2)}{\cos(\theta/2)} = 5. \tag{19}$$

Combining the existence of these one-sided end point limites with the continuity of the function h on the open interval $(0, 2\pi)$, we deduce that h is indeed bounded on the closed initerval $[0, 2\pi]$. By a classical theorem, it follows that h is Riemann-integrable on the interval $[0, 2\pi]$ and hence the sought-after integral is indeed well-defined.

We can now turn our attention to the computation of the integral. In order to compute this integral, we follow exactly the same procedure as in the preceeding exercise. This yields

$$\begin{aligned}
\int_0^{2\pi} \frac{\sin^2(5\theta/2)}{\sin^2(\theta/2)}\,\mathrm{d}\theta &= \int_0^{2\pi} \frac{\left(e^{i5\theta/2} - e^{-i5\theta/2}\right)^2}{\left(e^{i\theta/2} - e^{-i\theta/2}\right)^2}\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \left(\frac{e^{i5\theta/2} - e^{-i5\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}}\right)^2 \mathrm{d}\theta \\
&= \int_0^{2\pi} \left(\frac{e^{i5\theta} - 1}{e^{i\theta} - 1}\right)^2 \frac{e^{-5i\theta}}{e^{-i\theta}}\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \underbrace{-i\left(\frac{e^{i5\theta} - 1}{e^{i\theta} - 1}\right)^2 \frac{1}{e^{5i\theta}}}_{:=f(e^{i\theta})}\, ie^{i\theta}\mathrm{d}\theta = \int_\gamma f(z)\,\mathrm{d}z,
\end{aligned}$$

where γ is the unit circle parameterised by $\theta \mapsto e^{i\theta}$ for $\theta \in [0, 2\pi)$. We now study the function

$$f(z) = -i\left(\frac{z^5 - 1}{z - 1}\right)^2 \frac{1}{z^5} = -i\left(z^4 + z^3 + z^2 + z + 1\right)^2 \frac{1}{z^5}.$$

We deduce that the only singularity of the function f is located at $z = 0$, and this singularity is a pole of order five. Computing the residue at this singularity yields

$$\mathrm{Res}_0(f) = -\frac{i}{4!} \lim_{z\to 0} \frac{d^4}{dz^4}\left(z^4 + z^3 + z^2 + z + 1\right)^2 = -5i.$$

It therefore follows from the residue theorem that

$$\int_0^{2\pi} \frac{\sin^2(5\theta/2)}{\sin^2(\theta/2)}\, \mathrm{d}\theta = \int_\gamma f(z)\, \mathrm{d}z = 2\pi i\left(\mathrm{Res}_0(f)\right) = 10\pi.$$

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