MATH-207(d) Analysis IV
Exercise session 8

Exercice 1. Let $\gamma : [0,1] \to \mathbb{C}$ be a closed simple differentiable curve. Consider the functions

$$f(x) = e^{1/z}, \quad g(x) = e^{1/z^2}. \tag{1}$$

What are the possible values of the curve integrals of f and g over γ?

Answer. As a first step, we observe that the functions f and g are holomorphic on $\mathbb{C} \setminus \{0\}$ and have essential singularties at the origin $z_0 = 0$. Using, now our old trick of computing the Taylor expansion of $h(w) = \exp(w)$ and using the substitutions $w = 1/z$ and $w = 1/z^2$, we can write the Laurent series of f and g at $z_0 = 0$ as

$$f(z) = \sum_{n=-\infty}^{n=0} \frac{z^n}{(-n)!}, \qquad g(z) = \sum_{n=-\infty}^{n=0} \frac{z^{2n}}{(-n)!}.$$

It is now readily seen that

$$\mathrm{Res}_0(f) = 1, \qquad \mathrm{Res}_0(g) = 0.$$

Let U denote the open bounded set of $\mathbb{C}$ whose boundary is the curve γ. We now have three cases:

Case 1 If $0 \in U$, then by the residue theorem we have that

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i, \qquad \int_\gamma g(z)\,\mathrm{d}z = 0.$$

Case 2 If $0 \in \partial U = \gamma$, then

$$\int_\gamma f(z)\,\mathrm{d}z \text{ and } \int_\gamma g(z)\,\mathrm{d}z \text{ are both ill-defined.}$$

Case 3 If $0 \notin \overline{U}$, then by the Cauchy theorem we have that

$$\int_\gamma f(z)\,\mathrm{d}z = 0, \qquad \int_\gamma g(z)\,\mathrm{d}z = 0.$$

With that, the exercise is complete. ■

Exercice 2. Let $\alpha > 0$. Use the residue theorem to compute the following integrals

$$\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{1+x^2}\mathrm{d}x, \qquad \int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{1+x^2}\mathrm{d}x. \tag{2}$$

Hint: for simplicity, you can first try $\alpha = 1$. You can easily express the second integral in terms of the first integral.

Answer. First, there are different ways of expressing the second integral in terms of the first integral. Notice that for any $\alpha > 0$ and $x \in \mathbb{R}$, we can write

$$\int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{1+x^2}\mathrm{d}x = \int_{-\infty}^{\infty} \frac{\mathrm{Re}(e^{i\alpha x})}{1+x^2}\mathrm{d}x = \mathrm{Re}\Big(\underbrace{\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{1+x^2}\mathrm{d}x}_{:=(\mathrm{I})}\Big). \tag{3}$$

Therefore, it suffices to compute the improper integral (I). Alternatively, we could use

$$\int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{1+x^2}\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{i\alpha x}+e^{-i\alpha x}}{1+x^2}\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{1+x^2}\mathrm{d}x + \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{i(-\alpha)x}}{1+x^2}\mathrm{d}x, \tag{4}$$

but notice that now we need the last integral. One way of computing it is to replicate the discussion from the lecture but integrating over the lower half-plane[1]. Fortunately, we already know everything to circumvent that effort: for example, we can use the complex conjugate to find

$$\int_{-\infty}^{\infty} \frac{e^{i(-\alpha)x}}{1+x^2}\mathrm{d}x = \int_{-\infty}^{\infty} \frac{\overline{e^{i\alpha x}}}{1+x^2}\mathrm{d}x = \int_{-\infty}^{\infty} \overline{\frac{e^{i\alpha x}}{1+x^2}}\mathrm{d}x = \overline{\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{1+x^2}\mathrm{d}x}, \tag{5}$$

or we use a change of variables: with $y = -x$, we get

$$\int_{-\infty}^{\infty} \frac{e^{i(-\alpha)x}}{1+x^2}\mathrm{d}x = -\int_{\infty}^{-\infty} \frac{e^{i\alpha y}}{1+y^2}\mathrm{d}y = \int_{-\infty}^{\infty} \frac{e^{i\alpha y}}{1+y^2}\mathrm{d}y. \tag{6}$$

Any of those (simple!) tricks leads us to the same situation, one way or the other: we need to compute the integral (I).

With all that being settled, let's finally approach the integral (I). We can apply the example as seen in the lecture. Let us introduce the function

$$f(z) = \frac{e^{i\alpha z}}{1+z^2},$$

It is readily seen that f is holomorphic on $\mathbb{C} \setminus \{+i, -i\}$ and has simple poles at $z = +i$ and $z = -i$ with residues

$$\mathrm{Res}_{+i}(f) = \frac{e^{-\alpha}}{2i} \qquad \mathrm{Res}_{-i}(f) = \frac{e^{\alpha}}{2i}$$

As seen in the lecture, we have

$$\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{1+x^2}\mathrm{d}x = 2\pi i\,(\mathrm{Res}_{+i}(f)) = 2\pi i\left(\frac{e^{-\alpha}}{2i}\right) = \pi e^{-\alpha}. \tag{7}$$

More explicitly, let us first recall from the definition of improper integrals that we can write

$$(\mathrm{I}) = \lim_{L\to\infty} \int_{-L}^{L} \frac{e^{i\alpha x}}{1+x^2}\mathrm{d}x.$$

For simplicity, let us denote by γ_1 the open interval $(-L, L)$ viewed as a subset of $\mathbb{C}$, i.e., $\gamma_1 = \{z \in \mathbb{C}\colon \quad \mathrm{Im}(z) = 0,\ |z| < L\}$.

Consider now the curve $\gamma_2\colon [0, \pi] \to \mathbb{C}$ defined as $\gamma_2(t) = Le^{it}$. This curve is simply the semi-circle in the upper-half complex plane of radius L and center at the origin $z_0 = 0$. Consequently,

[1] This is an extra exercise if you find this all very easy.

$\gamma_1 \cup \gamma_2$ constitutes a closed simple regular curve. Applying now the residue theorem and making use of our knowledge of the poles of f, we can deduce that for any $L > 1$ it holds that

$$\int_{\gamma_1 \cup \gamma_2} f(z)\,\mathrm{d}z = 2\pi i \frac{e^{-\alpha}}{2i} = \pi e^{-\alpha}.$$

The trick now is to realise that for any $L > 1$, we have in fact that

$$\int_{\gamma_1 \cup \gamma_2} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z = \int_{-L}^{L} \frac{\cos(\alpha x)}{1+x^2}\mathrm{d}x + \int_{\gamma_2} f(z)\,\mathrm{d}z,$$

so that

$$\int_{-L}^{L} \frac{\cos(\alpha x)}{1+x^2}\mathrm{d}x = \pi e^{-\alpha} - \int_{\gamma_2} f(z)\,\mathrm{d}z, \tag{8}$$

provided, of course, that the later integral on the curve γ_2 is well-defined. We claim that this is indeed the case. To see this, observe that a direct calculation reveals that

$$\int_{\gamma_2} f(z)\,\mathrm{d}z = \int_0^{\pi} \frac{e^{i\alpha L e^{it}}}{1+(Le^{it})^2} iLe^{it}\,\mathrm{d}t. \tag{9}$$

Next, notice that for any $\alpha > 0$ there exists a constant $C_\alpha > 0$ depending only on α such that for all $t \in [0,\pi]$ and all $L > 1$ it holds that

$$|e^{i\alpha L e^{it}}| = |e^{i\alpha L\cos(t)}e^{-\alpha L \sin(t)}| = |e^{-\alpha L\sin(t)}| < C_\alpha.$$

Similarly, for any $\alpha > 0$, all $t \in [0,\pi]$ and all $L > 1$ it holds that

$$\frac{1}{|1+(Le^{it})^2|} = \frac{1}{|1+L^2e^{2it}|} = \frac{1}{|L^2e^{2it}-(-1)|} \leq \frac{1}{\big||L^2e^{2it}|-|-1|\big|} = \frac{1}{L^2-1}.$$

Returning now to the integral defined through Equation (9) we see that

$$\left|\int_{\gamma_2} f(z)\,\mathrm{d}z\right| \leq \int_0^{\pi} \left|\frac{e^{i\alpha L e^{it}}}{1+(Le^{it})^2} iLe^{it}\right|\,\mathrm{d}t \leq \int_0^{\pi} C_\alpha \frac{L}{L^2-1}\,\mathrm{d}t = C_\alpha \pi \frac{L}{L^2-1}, \tag{10}$$

and in particular

$$\lim_{L\to\infty} \int_{\gamma_2} f(z)\,\mathrm{d}z = \lim_{L\to\infty} C_\alpha \pi \frac{L}{L^2-1} = 0. \tag{11}$$

This finishes integral (I).
Finally, we can conclude that

$$\int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{1+x^2}\mathrm{d}x = \mathrm{Re}\Big(\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{1+x^2}\mathrm{d}x\Big) = \frac{\pi}{e^\alpha}. \tag{12}$$

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Exercice 3. Use the residue theorem to compute the integral

$$\int_{-\infty}^{\infty} \frac{e^{\alpha x i}}{4+x^4}\mathrm{d}x \tag{13}$$

Hint: The following facts might be useful:

$$z^2 = 2i \iff z = \pm(1+i), \qquad z^2 = -2i \iff z = \pm(1-i).$$

Answer. First, it suffices to consider the case $\alpha \geq 0$. For example, we can use the complex conjugate to find

$$\int_{-\infty}^{\infty} \frac{e^{i(-\alpha)x}}{4+x^4}\mathrm{d}x = \int_{-\infty}^{\infty} \frac{\overline{e^{i\alpha x}}}{4+x^4}\mathrm{d}x = \int_{-\infty}^{\infty} \overline{\frac{e^{i\alpha x}}{4+x^4}}\mathrm{d}x = \overline{\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{4+x^4}\mathrm{d}x}. \tag{14}$$

So let us first discuss the case $\alpha \geq 0$. We follow exactly the same strategy as in the previous exercise. Let us introduce the function

$$f(z) = \frac{e^{i\alpha z}}{4+z^4}.$$

The polynomial $4+z^4$ can be factorised as $4+z^4 = (z^2-2i)(z^2+2i)$. Using therefore the hint provided in this exercise we deduce that f is holomorphic on $\mathbb{C}\setminus S$ where the set S is defined as

$$S = \{1+i, -1-i, 1-i, -1+i\}.$$

Moreover, at each of the four points in S, the function f has a simple pole. The residues can be computed with some basic (though tedious) calculations and are given by

$$\mathrm{Res}_{1+i}(f) = \left(-\frac{1}{16} - \frac{i}{16}\right)e^{-\alpha+\alpha i}, \qquad \mathrm{Res}_{-1-i}(f) = \left(\frac{1}{16} + \frac{i}{16}\right)e^{\alpha-\alpha i},$$
$$\mathrm{Res}_{1-i}(f) = \left(-\frac{1}{16} + \frac{i}{16}\right)e^{\alpha+\alpha i}, \qquad \mathrm{Res}_{-1+i}(f) = \left(\frac{1}{16} - \frac{i}{16}\right)e^{-\alpha-\alpha i}.$$

We only need the ones in the upper half-plane. We observe

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{4+x^4}\mathrm{d}x &= 2\pi i\left(\mathrm{Res}_{1+i}(f) + \mathrm{Res}_{-1+i}(f)\right) \\
&= 2\pi i e^{-\alpha}\left(\left(-\frac{1}{16} - \frac{i}{16}\right)e^{\alpha i} + \left(\frac{1}{16} - \frac{i}{16}\right)e^{-\alpha i}\right) \\
&= 2\pi i e^{-\alpha}\left(\left(-\frac{1}{16} - \frac{i}{16}\right)(\cos(\alpha) + \sin(\alpha)i) + \left(\frac{1}{16} - \frac{i}{16}\right)(\cos(\alpha) - \sin(\alpha)i)\right) \\
&= \frac{1}{4}\pi e^{-\alpha}\left(\sin(\alpha) + \cos(\alpha)\right).
\end{aligned}$$

Up to now, we have considered the case $\alpha \geq 0$. If $\alpha < 0$, then we find

$$\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{4+x^4}\mathrm{d}x = \overline{\frac{1}{4}\pi e^{\alpha}\left(\sin(-\alpha) + \cos(-\alpha)\right)}. = \frac{1}{4}\pi e^{\alpha}\left(\sin(-\alpha) + \cos(-\alpha)\right). \tag{15}$$

In summary, for any $\alpha \in \mathbb{R}$, we get

$$\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{4+x^4}\mathrm{d}x = \overline{\frac{1}{4}\pi e^{\alpha}\left(\sin(-\alpha) + \cos(-\alpha)\right)}. = \frac{1}{4}\pi e^{-|\alpha|}\left(\sin(|\alpha|) + \cos(|\alpha|)\right). \tag{16}$$

We can also discuss this exercise more explicitly and replicate the discussion from the lecture. The goal is to compute

$$\lim_{L\to\infty} \int_{-L}^{L} \frac{e^{i\alpha x}}{4+x^4}\mathrm{d}x.$$

For simplicity, let us denote by γ_1 the open interval $(-L, L)$ viewed as a subset of $\mathbb{C}$, i.e. $\gamma_1 = \{z \in \mathbb{C}: \quad \mathrm{Im}(z) = 0,\ |z| < L\}$.

Consider once again the curve $\gamma_2 \colon [0, \pi] \to \mathbb{C}$ defined as $\gamma_2(t) = Le^{it}$. As in the previous exercise, this curve is the semi-circle in the upper-half complex plane of radius L and center at the origin $z_0 = 0$. Moreover, $\gamma_1 \cup \gamma_2$ constitutes a closed simple regular curve so that we can make use of our knowledge of the poles of f and apply the residue theorem. We thus obtain that for all $L > \sqrt{2}$ it holds that

$$\int_{\gamma_1 \cup \gamma_2} f(z) \, \mathrm{d}z = 2\pi i \left(\left(-\frac{1}{16} - \frac{i}{16} \right) e^{-\alpha + \alpha i} + \left(\frac{1}{16} - \frac{i}{16} \right) e^{-\alpha - \alpha i} \right)$$
$$= \frac{1}{4} \pi e^{-\alpha} \left(\sin(\alpha) + \cos(\alpha) \right).$$

As in the previous exercise, we notice that for any $L > \sqrt{2}$, we have that

$$\int_{\gamma_1 \cup \gamma_2} f(z) \, \mathrm{d}z = \int_{\gamma_1} f(z) \, \mathrm{d}z + \int_{\gamma_2} f(z) \, \mathrm{d}z = \int_{-L}^{L} \frac{e^{i\alpha x}}{4 + x^4} \mathrm{d}x + \int_{\gamma_2} f(z) \, \mathrm{d}z,$$

so that

$$\int_{-L}^{L} \frac{e^{i\alpha x}}{4 + x^4} \mathrm{d}x = \frac{1}{4} \pi e^{-\alpha} \left(\sin(\alpha) + \cos(\alpha) \right) - \int_{\gamma_2} f(z) \, \mathrm{d}z, \tag{17}$$

provided that the integral on the curve γ_2 is well-defined. To show that this is indeed the case, we again observe thatobserve that a direct calculation reveals that

$$\int_{\gamma_2} f(z) \, \mathrm{d}z = \int_{0}^{\pi} \frac{e^{i\alpha L e^{it}}}{4 + (Le^{it})^4} iLe^{it} \, \mathrm{d}t. \tag{18}$$

Arguing exactly as in the previous exercise, we deduce that for any $\alpha > 0$ there exists a constant $C_\alpha > 0$ depending only on α such that for all $t \in [0, \pi]$ and all $L > 1$ it holds that

$$|e^{i\alpha L e^{it}}| < C_\alpha.$$

Similarly, for any $\alpha > 0$, all $t \in [0, \pi]$ and all $L > \sqrt{2}$ it holds that

$$\frac{1}{|4 + (Le^{it})^4|} = \frac{1}{|4 + L^4 e^{4it}|} = \frac{1}{|L^4 e^{4it} - (-4)|} \leq \frac{1}{\big| |L^4 e^{4it}| - |-4| \big|} = \frac{1}{L^4 - 4}.$$

Returning now to the integral defined through Equation (18) we see that

$$\left| \int_{\gamma_2} f(z) \, \mathrm{d}z \right| \leq \int_{0}^{\pi} \left| \frac{e^{i\alpha L e^{it}}}{4 + (Le^{it})^4} iLe^{it} \right| \, \mathrm{d}t \leq \int_{0}^{\pi} C_\alpha \frac{L}{L^4 - 4} \, \mathrm{d}t = C_\alpha \pi \frac{L}{L^4 - 4}, \tag{19}$$

and in particular

$$\lim_{L \to \infty} \int_{\gamma_2} f(z) \, \mathrm{d}z = \lim_{L \to \infty} C_\alpha \pi \frac{L}{L^4 - 4} = 0. \tag{20}$$

Consequently, Equation (17) now yields that

$$\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{4 + x^4} \mathrm{d}x = \lim_{L \to \infty} \int_{-L}^{L} \frac{e^{i\alpha x}}{4 + x^4} \mathrm{d}x = \frac{1}{4} \pi e^{-\alpha} \left(\sin(\alpha) + \cos(\alpha) \right) - \lim_{L \to \infty} \int_{\gamma_2} f(z) \, \mathrm{d}z$$
$$= \frac{1}{4} \pi e^{-\alpha} \left(\sin(\alpha) + \cos(\alpha) \right).$$

Remark: Notice that this calculation yields a Fourier transform: if $f(x) = (4 + x^4)^{-1}$, then

$$\mathfrak{F}[f](\alpha) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{\alpha x i}}{4 + x^4} \mathrm{d}\alpha = \frac{1}{4\sqrt{2}} \sqrt{\pi} e^{-|\alpha|} \left(\sin(|\alpha|) + \cos(|\alpha|)\right).$$

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Exercice 4. Compute

$$\int_0^{2\pi} \frac{\cos^2\theta}{13 - 5\cos 2\theta} \mathrm{d}\theta. \tag{21}$$

Hint: Use the residue theorem by recasting this integral as a contour integral on the unit circle. The starting point is to observe that for $z = e^{i\theta}$ we have

$$\begin{aligned} \cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right) \\ \cos 2\theta &= \frac{e^{2i\theta} + e^{-2i\theta}}{2} = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right). \end{aligned}$$

Answer. We use hints and replicate the discussion as in the lecture: Then we have that

$$\begin{aligned} &\int_0^{2\pi} \frac{\cos^2\theta}{13 - 5\cos 2\theta} d\theta \\ &= \int_0^{2\pi} \frac{\frac{(e^{i\theta}+e^{-i\theta})^2}{4}}{13 - 5\frac{e^{2i\theta}+e^{-2i\theta}}{2}} \mathrm{d}\theta \\ &= \int_0^{2\pi} \frac{\frac{(e^{i\theta}+e^{-i\theta})^2}{4}}{13 - \frac{5}{2}\left(e^{2i\theta} + e^{-2i\theta}\right)} \mathrm{d}\theta \\ &= \int_0^{2\pi} \underbrace{\frac{\frac{(e^{i\theta}+e^{-i\theta})^2}{4}}{13 - \frac{5}{2}\left(e^{2i\theta} + e^{-2i\theta}\right)} \frac{1}{ie^{it}}}_{:=f(e^{it})} \cdot ie^{it} \mathrm{d}\theta \\ &= \int_\gamma f(z) \mathrm{d}z, \end{aligned}$$

where γ is the unit circle parametrized as $\theta \mapsto e^{i\theta}$, for $\theta \in [0, 2\pi]$. We study the function

$$f(z) = \frac{\frac{(z+\frac{1}{z})^2}{4}}{13 - \frac{5}{2}\left(z^2 + \frac{1}{z^2}\right)} \frac{1}{iz}.$$

We simplify it:

$$\begin{aligned} &f(z) \\ &\quad = \frac{1}{4i} \frac{(z + \frac{1}{z})^2}{13 - \frac{5}{2}\left(z^2 + \frac{1}{z^2}\right)} \frac{1}{z} \\ &\quad = \frac{1}{4i} \frac{(z + \frac{1}{z})^2}{13z - \frac{5}{2}\left(z^3 + \frac{1}{z}\right)} \\ &\quad = \frac{1}{4i} \frac{z(z + \frac{1}{z})^2}{13z^2 - \frac{5}{2}\left(z^4 + 1\right)}. \end{aligned}$$

In the numerator, we use

$$z(z+\frac{1}{z})^2 = z(z^2+2+\frac{1}{z^2}) = (z^3+2z+\frac{1}{z}) = \frac{z^4+2z^2+1}{z}.$$

Hence

$$f(z) = \frac{1}{4i}\frac{z^4+2z^2+1}{z(13z^2-\frac{5}{2}(z^4+1))} = \frac{1}{4i}\frac{2(z^4+2z^2+1)}{z(26z^2-5z^4-5)}.$$

We can apply the residue theorem if we find the singularities of f inside the unit circle. The singularities of f are $z=0$, $\pm 1/\sqrt{5}$ and $\pm\sqrt{5}$ and they are all pole of order 1. But only the first three are in inside the unit circle! We compute their residues, knowing that they are all poles of order 1:

$$\begin{aligned}
\mathrm{Res}_0(\widetilde{f}) &= \lim_{z\to 0} z\widetilde{f}(z) = \frac{-1}{10i}\\
\mathrm{Res}_{1/\sqrt{5}}(\widetilde{f}) &= \lim_{z\to 1/\sqrt{5}}\left(z-\frac{1}{\sqrt{5}}\right)\widetilde{f}(z) = \frac{3}{40i}\\
\mathrm{Res}_{-1/\sqrt{5}}(\widetilde{f}) &= \lim_{z\to -1/\sqrt{5}}\left(z+\frac{1}{\sqrt{5}}\right)\widetilde{f}(z) = \frac{3}{40i}.
\end{aligned}$$

Thus we find that

$$\int_0^{2\pi}\frac{\cos^2\theta}{13-5\cos 2\theta}\mathrm{d}\theta = 2\pi i\left(\mathrm{Re}_0(\tilde{f}) + \mathrm{Res}_{1/\sqrt{5}}(\tilde{f}) + \mathrm{Res}_{-1/\sqrt{5}}(\tilde{f})\right) = \frac{\pi}{10}.$$

The calculation is complete. ■

Exercice 5. Let $\gamma = \left\{z\in\mathbb{C} : |z-\frac{\pi}{2}| = 1\right\}$. Compute the value of

$$\int_\gamma \frac{z^2\sin(z)}{(z-\frac{\pi}{2})^2}\mathrm{d}z$$

with

(a) the Cauchy integral formula.

(b) the residue theorem.

Answer.

(a) We apply the Cauchy integral formula to $f(\xi) = \xi^2\sin(\xi)$, $z=\frac{\pi}{2}$ and $n=1$. We have $f'(z) = 2z\sin(z) + z^2\cos(z)$. From which we find

$$\int_\gamma \frac{z^2\sin(z)}{(z-\frac{\pi}{2})^2}\mathrm{d}z = \frac{2\pi i}{1!}f'\left(\frac{\pi}{2}\right) = 2\pi^2 i.$$

(b) Denoting $g(z) = \frac{z^2\sin(z)}{(z-\frac{\pi}{2})^2}$, by the residue theorem we know that

$$\int_\gamma \frac{z^2\sin(z)}{(z-\frac{\pi}{2})^2}\mathrm{d}z = 2\pi i\mathrm{Res}_{\frac{\pi}{2}}(g).$$

Since the numerator of g is not zero in $\frac{\pi}{2}$ and the denominator features a zero of order 2 in $\frac{\pi}{2}$ then we deduce $z=\frac{\pi}{2}$ is a pole of order $2-0=2$. Thus we have

$$\mathrm{Res}_{\frac{\pi}{2}}(g) = \frac{1}{(2-1)!}\lim_{z\to\frac{\pi}{2}}\left\{\frac{\mathrm{d}^{2-1}}{\mathrm{d}z^{2-1}}\left[\left(z-\frac{\pi}{2}\right)^2 g(z)\right]\right\} = \lim_{z\to\frac{\pi}{2}} f'(z) = f'\left(\frac{\pi}{2}\right) = \pi. \quad (22)$$

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