MATH-207(d) Analysis IV
Exercise session 7

Exercice 1. Let $\gamma(t) = e^{it}$ be the standard parameterization of the unit circle. Compute the following integrals

$$A = \int_\gamma \frac{e^z}{z^2(z-2)} \, dz, \qquad B = \int_\gamma \frac{\sin(z)}{z(z+2i)} \, dz,$$
$$C = \int_\gamma \frac{z^3 - iz}{z(z-2i)} \, dz, \qquad D = \int_\gamma \frac{1}{z} - \frac{1}{z^3} + z\sin(z)e^x \, dz.$$

Answer.

(a) We use the Cauchy integral formula, with $n = 1$:

$$A = \int_\gamma \frac{e^z}{z^2(z-2)} \, dz = \int_\gamma \frac{e^z/(z-2)}{z^2} \, dz \tag{1}$$
$$= 2\pi i \cdot \left(\frac{e^z}{z-2}\right)'(0) = 2\pi i \cdot \left(\frac{e^z(z-2) - e^z}{(z-2)^2}\right)(0) = 2\pi i \cdot \left(\frac{-3}{4}\right) = \frac{-3}{2}\pi i. \tag{2}$$

(b) Again, we use the Cauchy integral formula, with $n = 0$:

$$B = \int_\gamma \frac{\sin(z)}{z(z+2i)} \, dz \tag{3}$$
$$= \int_\gamma \frac{\sin(z)/(z+2i)}{z} \, dz = 2\pi i \frac{\sin(0)}{0-2i} = 0. \tag{4}$$

(c) Similar as above,

$$C = \int_\gamma \frac{z^3 - iz}{z(z-2i)} \, dz \tag{5}$$
$$= \int_\gamma \frac{(z^3 - iz)/(z-2i)}{z} \, dz = (0^3 - i0)/(z-2i) = 0. \tag{6}$$

Note that $z = 0$ is a regular point of the integrand, as can be seen by the powers in the numerator and the denominator.

(d) The integral D is easy:

$$D = \int_\gamma \frac{1}{z} \, dz - \int_\gamma \frac{1}{z^3} \, dz + \int_\gamma z\sin(z)e^x \, dz = \int_\gamma \frac{1}{z} \, dz = 2\pi i. \tag{7}$$

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Exercice 2. Let $\gamma(t) = 10e^{it}$ be a parameterization of the circle around 0 with radius 10. Consider the function

$$f(z) = \frac{\sin(z)}{(z+1)(z+2)(z+3)}. \tag{8}$$

What are the singularities of f and what the orders of the poles? Calculate

$$\int_\gamma f(z)\mathrm{d}z. \tag{9}$$

Hint: express the numerator and denominator as Taylor series at different points; you don't need all the coefficients. To compute the residues, you can either use the formula from the lecture or the Cauchy integral formulas. Try both!

Answer. The singularities are the zeroes of the denominator: $z_1 = -1$, $z_2 = -2$, and $z_3 = -3$. They all lie within the circle γ, and hence we can use the residue theorem:

$$\int_\gamma f(z)\mathrm{d}z = 2\pi i \cdot \Big(\mathrm{Res}_{-1}(f) + \mathrm{Res}_{-2}(f) + \mathrm{Res}_{-3}(f)\Big). \tag{10}$$

It remains to determine the residues at those singularities.

First, we determine the order of the poles. We know that $\sin(-1)$, $\sin(-2)$, and $\sin(-3)$ are non-zero, so the Taylor series of sine at 1, 2, and 3 must have a non-zero constant term. Since the factors $(z+1)$, $(z+2)$, and $(z+3)$ only appear once in the denominator, they are poles of order 1. Formally, we can also verify this with the discussion as seen in the lecture.
For example, if $\sin(z) = c_0 + c_1(z+1) + c_2(z+1)^2 + \dots$ is the Taylor series of sin at 1, then

$$\frac{\sin(z)}{(z+1)(z+2)(z+3)} = \frac{c_0 + c_1(z+1) + c_2(z+1)^2 + \dots}{(z+1)(z^2+5z+6)} \tag{11}$$

$$= \frac{c_0 + c_1(z+1) + c_2(z+1)^2 + \dots}{(z+1)(12+7(z-1)+(z-1)^2)} \tag{12}$$

$$= \frac{c_0 + c_1(z+1) + c_2(z+1)^2 + \dots}{12(z+1)+7(z-1)^2+(z-1)^3}. \tag{13}$$

Another way of seeing the same thing notes that

$$\frac{\sin(z)/(z+2)(z+3)}{(z+1)} \tag{14}$$

has a holomorphic numerator near -1 (with non-zero constant term) and a denominator with Taylor series having only the first power.

Second, as these are first-order poles, we need to compute the residuals. We may use the formula from the lecture and obtain

$$\mathrm{Res}_{-1}(f) = \lim_{z\to-1}(z+1)f(z) = \lim_{z\to-1}\frac{\sin(z)}{(z+2)(z+3)} = \frac{\sin(-1)}{2}, \tag{15}$$

$$\mathrm{Res}_{-2}(f) = \lim_{z\to-2}(z+2)f(z) = \lim_{z\to-2}\frac{\sin(z)}{(z+1)(z+3)} = \frac{\sin(-2)}{-1} = -\sin(-2), \tag{16}$$

$$\mathrm{Res}_{-3}(f) = \lim_{z\to-3}(z+3)f(z) = \frac{\sin(-3)}{2}. \tag{17}$$

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Exercice 3 (Essential singularities)**.** Consider the function $f(z) = e^{1/z}$.

(a) Determine the nature of the singularity.

(b) Let $z_0 = x_0 + iy_0$ be a point on the complex unit sphere. Study the value $f(tz_0)$ as t moves closer to zero depending on z_0. *Hint: in other words, we study how f behaves as we move straight towards the origin from different directions.*

(c) Given $y \in \mathbb{C}$, identify all the $w \in \mathbb{C}$ for which $y = e^w$.

(d) Show that in every small neighborhood of f, every complex number is attained by f infinitely often. Formally: show that for every $R > 0$ and $y \in \mathbb{C}$ there exist infinitely many $z \in B_R(0)$ for which $y = f(z)$.

Answer.

(a) As seen in the lecture (see also previous Exercise Sheet 6), this function has an essential singularity at the origin.

(b) Let $z = x + iy \in \mathbb{C}$ be arbitrary. Obviously, it holds that

$$f(z) = e^{1/z} = e^{\frac{x-yi}{x^2+y^2}} = e^{\frac{x}{x^2+y^2}}\, e^{\frac{-yi}{x^2+y^2}}.$$

Consider now any point $z_0 = x_0 + iy_0$ on the complex unit sphere. Then for any $t \in \mathbb{R}$, we have that $tz_0 = tx_0 + ity_0$ so that

$$f(tz_0) = e^{\frac{tx_0}{t^2x_0^2+t^2y_0^2}}\, e^{\frac{-ty_0i}{t^2x_0^2+t^2y_0^2}} = e^{\frac{1}{t}\cdot\frac{x_0}{x_0^2+y_0^2}}\, e^{\frac{1}{t}\cdot\frac{-y_0i}{x_0^2+y_0^2}} = e^{\frac{x_0}{t}}\, e^{\frac{-y_0}{t}i}.$$

In order to build some intuition about the behaviour of the function $f(tz_0)$ as $t \to 0$, let us first consider two simple cases:

- In the case $x_0 = 0$, we have $|y_0| = 1$ and thus

$$f(z) = e^{i\frac{-y_0}{t}} = e^{\frac{\pm i}{t}}. \tag{18}$$

  Consequently, as t goes to zero, $f(z)$ will traverse the unit circle. Depending on whether $y_0 = 1$ or $y_0 = -1$, it will circle in the counter clockwise or clockwise direction.

- In the case $y_0 = 0$, we have $|x_0| = 1$ and thus

$$f(z) = e^{\frac{x_0}{t}} = e^{\frac{\pm 1}{t}}. \tag{19}$$

  Consequently, as t goes to zero, $f(z)$ will go to either real $+\infty$ or 0, depending on whether $x_0 = 1$ or $x_0 = -1$.

Equipped with this intuition, we can now return to the general case and recall that

$$f(tz_0) = e^{\frac{x_0}{t}}\, e^{\frac{-y_0}{t}i}, \tag{20}$$

where $z_0 = x_0 + iy_0$. The behavior of the function $f(tz_0)$ for $t \to 0$ is now clear:

(i) Depending on whether x_0 is positive or negative, $f(tz_0)$ will either go to zero towards infinity as t goes to zero. Additionally, the convergence will be slow for x_0 close to zero.

(ii) Depending on whether y_0 is positive or negative, $f(z_0)$ will circle clockwise or counter clockwise around the unit circle as t goes to zero.

(c) Consider some arbitrary $0 \neq y \in \mathbb{C}$ with polar representation

$$y = re^{i\theta}, \quad \text{for some } \ r > 0 \text{ and } \theta \in [0, 2\pi). \tag{21}$$

Let $w = a + ib \in \mathbb{C}$ with $a, b \in \mathbb{R}$. Then it follows that

$$e^w = e^a e^{ib}.$$

Since we seek $w \in \mathbb{C}$ such that $e^w = y$, we must have that

$$r = e^a \quad \Longrightarrow \quad a = \log(r).$$

This in turn implies that we must also have

$$b = \theta + 2\pi n \quad \text{for any } n \in \mathbb{Z}.$$

Consequently, for an arbitrary $y \in \mathbb{C}$ with polar representation $y = re^{i\theta}$, the set of all $w \in \mathbb{C}$ such that $y = e^w$ is given by

$$w = \log(r) + i\left(\theta + 2\pi n\right).$$

In particular, we see that the function $w \mapsto e^w$ attains every value $y \neq 0$ for infinitely many choices of w.

(d) We want to study the values of the function $e^{1/z}$ over the set

$$\dot{B}_R(0) = \left\{z \in \mathbb{C} \ : \ 0 < |z| < R\right\},$$

which is the complex *punctured disk* of radius $R > 0$. As a first remark, we observe that the function $z \mapsto 1/z$ defines a bijection from $\dot{B}_R(0)$ to the set $\mathbb{C} \setminus \overline{B_{1/R}(0)}$. Explicitly,

$$\mathbb{C} \setminus \overline{B_{1/R}(0)} = \left\{z \in \mathbb{C} \ : \ R < |z|\right\}$$

We want to show that there are infinitely many $z \in \mathbb{C}$ with $0 < |z| < R$ such that $y = e^{1/z}$, no matter how small $R > 0$. Equivalently, we can also show that there are infinitely many $w \in \mathbb{C}$ with $|w| > 1/R$ such that $y = e^w$.

Consider some arbitrary $0 \neq y \in \mathbb{C}$ with polar representation

$$y = re^{i\theta}, \quad \text{for some } \ r > 0 \text{ and } \theta \in [0, 2\pi).$$

It follows that from the previous sub-task of the present exercise that $y = e^w$ for all $w \in \mathbb{C}$ of the form

$$w = \log(r) + i\left(\theta + 2\pi n\right).$$

In particular, for any choice of $R > 0$, no matter how small, there exists $n_0 \in \mathbb{Z}$ such that for all $n \geq n_0$

$$|w|^2 = |\log(r)|^2 + |\theta + 2\pi n|^2 > \frac{1}{R^2}.$$

This completes the proof.

Remark: this behavior of $e^{\frac{1}{z}}$ *is a good example of how complex functions behave near essential singularities: near this essential singularity, every complex number except zero is attained infintely often, no matter how close we "zoom in" towards the singularity. In that sense, the essential singularity behaves "chaotically". This is also known as the* Great Picard Theorem. ■

Exercice 4. Consider the function $f(z) = \frac{1}{z^4-1}$.

(a) Determine the singularities of f and their nature.

(b) Let γ be a circle of radius $r > 0$ centered at the origin. Determine the values of the integral

$$\int_\gamma f(z)\,\mathrm{d}z \tag{22}$$

for cases $r = 0.5$, $r = 1$, and $r = 2$.

(c) More generally, determine the integral for any $r > 0$.

Answer.

(a) Notice that we can write

$$f(z) = \frac{1}{z^4-1} = \frac{1}{\left(z^2-1\right)\left(z^2+1\right)} = \frac{1}{(z+1)(z-1)(z+i)(z-i)}.$$

Consequently, then function f has a simple pole at the points $z = \pm 1$ and $z = \pm i$.

(b) We consider each case seperately.

Case $r = 0.5$ In this case, the function f is analytic on $\overline{\mathrm{int}\gamma}$. Thus, by Cauchy's theorem, we have that

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Case $r = 1$ In this case, the function f has four singularities on the curve γ. Consequently, the integral is ill-defined.

Case $r = 2$ In this case, the function f has four simple poles in $\mathrm{int}\gamma$. Thus, the residue theorem implies that

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i\left(\mathrm{Re}_{-1} + \mathrm{Re}_1 + \mathrm{Re}_{-i} + \mathrm{Re}_i\right).$$

Computing each of the above residues yields

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i\left(-\frac{1}{4} + \frac{1}{4} - \frac{i}{4} + \frac{i}{4}\right) = 0.$$

(c) The general case follows easily from similar arguments as above, Indeed, we have that

$$\int_\gamma f(z)\,\mathrm{d}z = \begin{cases} 0 & \text{if } r \neq 1 \\ \text{ill-defined} & \text{if } r = 1 \end{cases}$$

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Exercice 5. Let $\gamma \subset \mathbb{C}$ be any simple close piecewise regular curve. Compute the following integrals depending on the curve γ.

(a) $\int_\gamma e^{1/z^2} \mathrm{d}z$

(b) $\int_\gamma \frac{z^2+2z+1}{(z-3)^3} \mathrm{d}z$

(c) $\int_\gamma \frac{e^{1/z}}{z^2} \mathrm{d}z$

(d) $\int_\gamma \frac{1}{(z-i)(z+2)^2(z-4)} \mathrm{d}z$

(e) $\int_\gamma \frac{\sin(z)}{z} \mathrm{d}z$

Answer.

(a) Case 1: $0 \in \mathrm{int}(\gamma)$. We compute the Laurent series in $z = 0$

$$e^{1/z^2} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z^2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{1}{z^{2n}} + 1.$$

We see that $\mathrm{Res}_0(f) = 0$ and therefore

$$\int_\gamma f(z) \mathrm{d}z = 0.$$

NB: the function is not holomorphic on the interior of γ, but the integral is zero nonetheless.

Case 2: $0 \notin \overline{\mathrm{int}(\gamma)}$. By Cauchy theorem we immediately conclude $\int_\gamma f(z) \mathrm{d}z = 0$.

Case 3: $0 \in \gamma$. The integral is ill-defined as the curve passes through the singularity.

(b) Denote $f(z) = \frac{z^2+2z+1}{(z-3)^3}$.

First observe that since the numerator is not zero in $z = 3$, we know that 3 is a pole of order 3. Then the residue is

$$\mathrm{Res}_3(f) = \frac{1}{2} \lim_{z \to 3} \frac{d^2}{dz^2} \left(z^2 + 2z + 1\right) = 1.$$

We now distinguish three cases.

Case 1: $3 \notin \overline{\mathrm{int}(\gamma)}$. By Cauchy theorem we can immediately conclude

$$\int_\gamma f(z) \mathrm{d}z = 0.$$

Case 2 : $3 \in \mathrm{int}\gamma$.

$$\int_\gamma f(z)\mathrm{d}z = 2\pi i \operatorname{Re}_3(f) = 2\pi i.$$

Case 3: $3 \in \gamma$. The integral is ill-defined.

(c) Denote $f(z) = \frac{e^{1/z}}{z^2}$.

The Laurent series in $z = 0$ is

$$f(z) = \frac{e^{1/z}}{z^2} = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{z^n} = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{z^{n+2}},$$

Therefore $z = 0$ is an isolated essential singularity and we have

$$\operatorname{Res}_0(f) = 0.$$

Case 1: $0 \notin \overline{\operatorname{int}(\gamma)}$. By Cauchy theorem we immediately conclude

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

Case 2: $0 \in \operatorname{int}(\gamma)$.

$$\int_\gamma f(z)\mathrm{d}z = 2\pi i \operatorname{Re}_0(f) = 0.$$

Case 3: $0 \in \gamma$. The integral is ill-defined.

(d) Denote

$$f(z) = \frac{1}{(z-i)(z+2)^2(z-4)}.$$

We start by computing the residues in $i, 4$ and -2. The first two are poles of order 1 whereas -2 is a pole of order 2.

$$\begin{aligned}
\operatorname{Res}_i(f) &= \lim_{z\to i}(z-i)f(z) = \frac{1}{(i+2)^2(i-4)} \\
\operatorname{Res}_4(f) &= \lim_{z\to 4}(z-4)f(z) = \frac{1}{36(4-i)} \\
\operatorname{Res}_{-2}(f) &= \lim_{z\to -2}\frac{\mathrm{d}}{\mathrm{d}z}\left[(z+2)^2 f(z)\right] = \lim_{z\to -2}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{(z-i)(z-4)}\right] \\
&= \lim_{z\to -2}\left(\frac{-2z+4+i}{(z-i)^2(z-4)^2}\right) = \frac{8+i}{36(i+2)^2}.
\end{aligned}$$

We now distinguish several cases.

Case 1: $i, -2, 4 \notin \operatorname{int}(\gamma)$. Then by Cauchy theorem we immediately get

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

Case 2: Exactly one point among $i, -2, 4$ in is $\operatorname{int}(\gamma)$.

Subcase 2a: $i \in \operatorname{int}(\gamma)$ but $-2, 4 \notin \overline{\operatorname{int}(\gamma)}$.

$$\int_\gamma f(z)\mathrm{d}z = 2\pi i \operatorname{Res}_i(f) = \frac{2\pi i}{(i+2)^2(i-4)}.$$

Subcase 2b: $-2 \in \operatorname{int}(\gamma)$ but $i, 4 \notin \overline{\operatorname{int}(\gamma)}$.

$$\int_\gamma f(z)\mathrm{d}z = 2\pi i \operatorname{Res}_{-2}(f) = \frac{\pi i(8+i)}{18(i+2)^2}.$$

Subcase 2c: $4 \in \operatorname{int}(\gamma)$ but $i, -2 \notin \overline{\operatorname{int}(\gamma)}$.

$$\int_\gamma f(z)\mathrm{d}z = 2\pi i \operatorname{Res}_4(f) = \frac{\pi i}{18(4-i)}.$$

Case 3: Exact two points amongs $i, -2, 4$ is in $\operatorname{int}(\gamma)$.

Subcase 3a: $i, -2 \in \operatorname{int}(\gamma)$ but $4 \notin \overline{\operatorname{int}\gamma}$.

$$\int_\gamma f(z)\mathrm{d}z = 2\pi i \left(\operatorname{Res}_i(f) + \operatorname{Res}_{-2}(f)\right) = \frac{2\pi i}{(i+2)^2}\left(\frac{1}{i-4} + \frac{8+i}{36}\right).$$

Subcase 3b: $i, 4 \in \operatorname{int}\gamma$ but $-2 \notin \overline{\operatorname{int}\gamma}$.

$$\int_\gamma f(z)\mathrm{d}z = 2\pi i \left(\operatorname{Re}_i(f) + \operatorname{Re}_4(f)\right) = \frac{2\pi i}{(i-4)}\left(\frac{1}{(i+2)^2} - \frac{1}{36}\right).$$

Subcase 3c: $-2, 4 \in \operatorname{int}(\gamma)$ but $i \notin \overline{\operatorname{int}(\gamma)}$.

$$\int_\gamma f(z)\mathrm{d}z = 2\pi i \left(\operatorname{Res}_{-2}(f) + \operatorname{Re}_4(f)\right) = \frac{\pi i}{18}\left(\frac{8+i}{(i+2)^2} - \frac{1}{(i-4)}\right).$$

Case 4: $i, -2, 4 \in \operatorname{int}(\gamma)$.

$$\begin{aligned}\int_\gamma f(z)\mathrm{d}z &= 2\pi i \left(\operatorname{Res}_i(f) + \operatorname{Res}_2(f) + \operatorname{Res}_4(f)\right) \\ &= 2\pi i \left(\frac{1}{(i-4)(i+2)^2} + \frac{8+i}{36(i+2)^2} - \frac{1}{36(i-4)}\right) = 0.\end{aligned}$$

Case 5: $i \in \gamma$ or $-2 \in \gamma$ or $4 \in \gamma$. The integral is ill-defined.

(e) Consider the function $\mathbb{C} \ni z \mapsto f(z) = \sin(z)/z$. Clearly, this function is holomorphic on the punctured complex plane $\mathbb{C} \setminus \{0\}$. A priori, f appears to have a singularity at $z = 0$ but in order to study this singularity more carefully, let us compute the Laurent series of f at $z_0 = 0$. Using similar arguments as those utilised in Exercise Sheet 5 and 6, we deduce that

$$f(z) = \frac{\sin(z)}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!},$$

and the radius of convergece of this Laurent series is infinite. It follows that f has a so-called *removable* singularity at $z = 0$. In other words, by using the above Laurent series and redefining f at the point $z = 0$ as $f(0) = 1$, we obtain a *holomorphic extension* $\widetilde{f}$ of f over the entire complex plane, i.e.,

$$\widetilde{f}(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!} = \begin{cases} f(z) & \text{if } z \neq 0, \\ 1 & \text{if } z = 0. \end{cases}$$

Applying now Cauchy's theorem we deduce that

$$\int_\gamma f(z) \, \mathrm{d}z = \int_\gamma \widetilde{f}(z) \, \mathrm{d}z = 0,$$

for all curves γ such that $0 \notin \gamma$.

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Exercice 6. Determine whether f has a singularity at $z_0 = 0$ and if yes, determine the order of the pole.

(a) $f(z) = \frac{z^2+2z+3}{z+1}$

(b) $f(z) = \frac{1+i}{z^2+z}$

(c) $f(z) = \frac{z^3}{z^2-z}$

(d) $f(z) = \frac{z^{-2}+z^{-1}+1}{z^{-2}+z+4z^2}$

(e) $f(z) = \frac{z^{-5}+z^{-2}+z^2}{z^{-2}+z+4z^2}$

(f) $f(z) = \frac{z^{-1}+1}{z^{-4}+3}$

(g) $f(z) = \frac{z^{-7}+1}{1+z}$

Answer.

(a) Since $\lim_{z\to 0} f(z) = 3$, f does not have a singularity at $z_0 = 0$.

(b) In this case, f has a pole of order one at $z_0 = 0$. Indeed, we can write

$$f(z) = \frac{1}{z}\left(\frac{1+i}{z+1}\right) \qquad \text{where } g(z) = \frac{1+i}{z+1} \text{ is continuous at } z_0 = 0.$$

(c) In this case, we can write for all $z \neq 0$ that

$$f(z) = \frac{z^3}{z^2 - z} = \frac{z^2}{z-1}.$$

Thus, $\lim_{z\to 0} f(z) = 0$. Consequently, f has a removable singularity at $z_0 = 0$.

(d) For all $z \neq 0$, we can write

$$f(z) = \frac{z^{-2} + z^{-1} + 1}{z^{-2} + z + 4z^2} = \frac{1 + z + z^2}{1 + z^3 + 4z^4}.$$

Thus, $\lim_{z\to 0} f(z) = 1$. Consequently, f has a removable singularity at $z_0 = 0$.

(e) For all $z \neq 0$, we can write

$$f(z) = \frac{z^{-5} + z^{-2} + z^2}{z^{-2} + z + 4z^2} = \frac{1 + z^3 + z^7}{z^3 + z^6 + 4z^7} = \frac{1}{z^3}\left(\frac{1 + z^3 + z^7}{1 + z^3 + 4z^4}\right).$$

Since the function

$$g(z) = \frac{1 + z^3 + z^7}{1 + z^3 + 4z^4}$$

is continuous at $z_0 = 0$, we deduce that f has a third order pole at $z_0 = 0$.

(f) For all $z \neq 0$, we can write

$$f(z) = \frac{z^{-1} + 1}{z^{-4} + 3} = \frac{z^3 + z^4}{1 + 3z^4}.$$

Thus, $\lim_{z\to 0} f(z) = 0$. Consequently, f has a removable singularity at $z_0 = 0$.

(g) For all $z \neq 0$, we can write

$$f(z) = \frac{z^{-7} + 1}{1 + z} = \frac{1 + z^7}{z^7 + z^8} = \frac{1}{z^7}\left(\frac{1 + z^7}{1 + z}\right).$$

Since the function

$$g(z) = \frac{1 + z^7}{1 + z}$$

is continuous at $z_0 = 0$, we deduce that f has a seventh order pole at $z_0 = 0$.

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