MATH-207(d) Analysis IV
Exercise session 1

1. Complex numbers algebra. Compute the cartesian representation.

(a) $z_1 = (2+i)i$

(b) $z_2 = (3-i)(4+i)$

(c) $z_3 = i(1+i)(1-i)$

(d) $z_4 = (2-i)^3$

(e) $z_5 = \frac{1}{i}$

(f) $z_6 = \frac{5-i}{i}$

(g) $z_7 = \frac{4+i}{3+2i}$

(h) $z_8 = \frac{2-i}{1-i}$

(i) $z_9 = \frac{3-2i}{2-i}$

(j) $z_{10} = (1-4i)^{-2}$

Answer.

(a) $z_1 = -1+2i$.

(b) $z_2 = 13-i$.

(c) $z_3 = 2i$.

(d) $z_4 = 2-11i$.

(e) $z_5 = -i$.

(f) $z_6 = -1-5i$.

(g) $z_7 = \frac{(4+i)(3-2i)}{13} = \frac{14}{13} - \frac{5}{13}i$.

(h) $z_8 = \frac{3}{2} + \frac{1}{2}i$.

(i) $z_9 = \frac{8}{5} - \frac{1}{5}i$.

(j) $z_{10} = \frac{1}{-15-8i} = -\frac{15}{289} + \frac{8}{289}i$.

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2. Complex number powers. Let us consider sequences of complex numbers of the form $z_n = z_0^{n+1}$, for every $n \in \mathbb{N}$ and some $z_0 \in \mathbb{C}$. Describe (with pictures or words) the aspect of the sequence of points in the complex plane for the following choices of z_0.

(a) $z_0 = i$

(b) $z_0 = \rho_0 e^{i\theta_0}$ with $\rho_0 = 0.99$ and $\theta_0 = \frac{91\pi}{180}$.

(c) $z_0 = \rho_0 e^{i\theta_0}$ with $\rho_0 = 0.99$ and $\theta_0 = \frac{89\pi}{180}$.

(d) $z_0 = \rho_0 e^{i\theta_0}$ with $\rho_0 = 0.99$ and $\theta_0 = \frac{61\pi}{180}$.

(e) $z_0 = \rho_0 e^{i\theta_0}$ with $\rho_0 = 0.99$ and $\theta_0 = (3 - \sqrt{5})\pi \simeq \frac{137.5078\pi}{180}$.
You may need the help of a computer for this one! See also https://en.wikipedia.org/wiki/Golden_angle.

Answer.

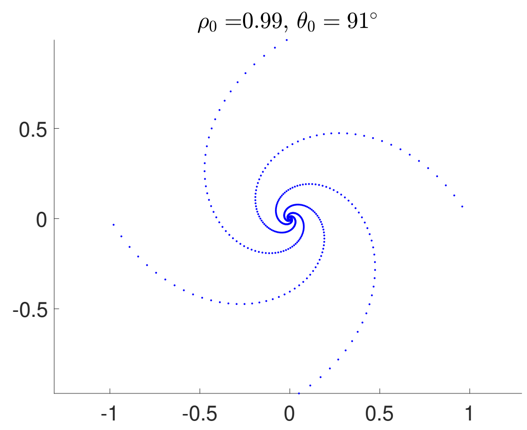
(a) Let's compute the first terms:

$$z_0 = i, \quad z_1 = -1, \quad z_2 = -i, \quad z_3 = 1, \quad z_4 = i.$$

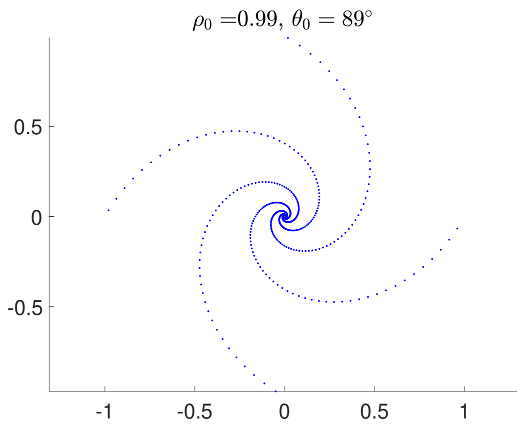
We notice the sequence is periodic with period 4.

$$z_n = \begin{cases} i & \text{if } n = 4k, \\ -1 & \text{if } n = 4k+1, \\ -i & \text{if } n = 4k+2, \\ 1 & \text{if } n = 4k+3. \end{cases}$$

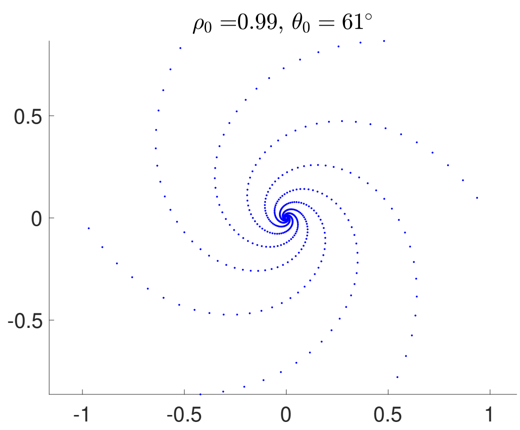
(b) This sequence will generate four spirals turning counterclockwise.



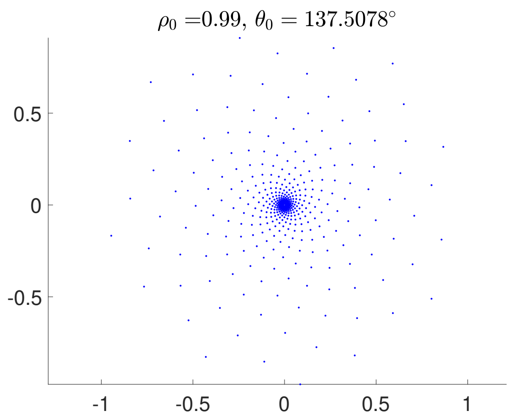
(c) This sequence generates four spirals turning clockwise.



(d) This sequence generates six spirals turning counterclockwise.



(e) This sequence generates a beautiful pattern often found in nature.



These images were generate with the following MATLAB code you can play with.

```
clc
close all;

% define parameters
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rho0 = 0.99;
% theta0 = 91 * pi / 180;
% theta0 = 89 * pi / 180;
% theta0 = 61 * pi / 180;
theta0 = (3-sqrt(5))*pi;

% define sequence
z0 = rho0 *(cos(theta0) + 1i * sin(theta0));
z =@(n) z0^n;

% plot the sequence
numberOfPointsToPlot = 500;

figure(1)
title(['$\rho_0 = $',num2str(rho0),', $\theta_0 ...
    =',num2str(theta0*180/pi),'^\circ$'],...
'interpreter','latex'), set(gca,'FontSize', 22),
axis equal, hold on,
set(gcf,'units','normalized','OuterPosition',[.5 .3 .5 .7])

for n = 1:numberOfPointsToPlot
zn = z(n);
plot(real(zn), imag(zn), 'b.');
%pause(1e-10); %uncomment for animation
end
hold off
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3. Euler's formula.

(a) For $z \in \mathbb{R}$, explain by pictures why

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \tag{1}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}. \tag{2}$$

(b) For $\alpha, \beta \in \mathbb{R}$, using that $e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)}$ show that

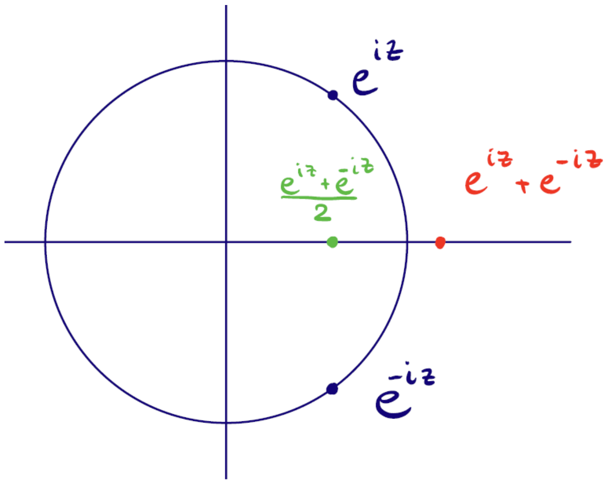
$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta),$$
$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta).$$

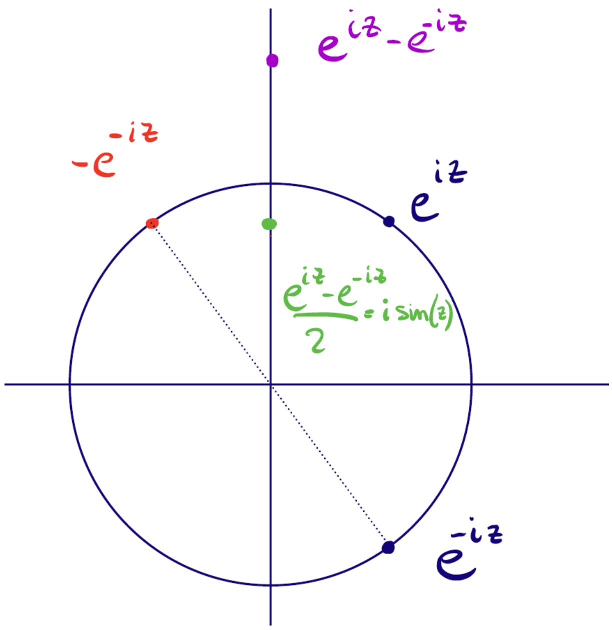
(c) We now extend Equations (1) and (2) to arbitrary complex numbers $z \in \mathbb{C}$. Show that these equations lead to the formula

$$\forall z \in \mathbb{C}: \quad e^{iz} = \cos(z) + i\sin(z).$$

Answer.

(a) Here are two constructions that show these formulas. As a starting point, we have that if $z \in \mathbb{R}$, e^{iz} is the complex number on the unit circle at oriented angle $\theta = z$. Then, scaling and adding such points works as with vectors of $\mathbb{R}^2$.





(b) We develop both sides:

$$\begin{aligned} e^{i\alpha}e^{i\beta} &= (\cos(\alpha) + i\sin(\alpha))(\cos(\beta) + i\sin(\beta)) \\ &= (\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)) + i(\cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta)). \\ e^{i(\alpha+\beta)} &= \cos(\alpha+\beta) + i\sin(\alpha+\beta) \end{aligned}$$

We then conclude by identifying real and imaginary parts: that is $\mathrm{Re}(e^{i\alpha}e^{i\beta}) = \mathrm{Re}(e^{i(\alpha+\beta)})$ and $\mathrm{Im}(e^{i\alpha}e^{i\beta}) = \mathrm{Im}(e^{i(\alpha+\beta)})$.

(c) Recall the Taylor expansion of the exponential of a real number x:

$$e^x = \sum_{k=0}^{+\infty} \frac{x^k}{k!}.$$

If we evaluate this expression in iz for any $z \in \mathbb{C}$:

$$\begin{aligned} e^{iz} = \sum_{k=0}^{+\infty} \frac{(iz)^k}{k!} &= 1 + iz + \frac{i^2 z^2}{2!} + \frac{i^3 z^3}{3!} + \frac{i^4 z^4}{4!} + \frac{i^5 z^5}{5!} + \dots \\ &= 1 + iz + i^2 \frac{z^2}{2!} + i^3 \frac{z^3}{3!} + i^4 \frac{z^4}{4!} + i^5 \frac{z^5}{5!} + \dots \\ &= 1 + iz - \frac{z^2}{2!} - i\frac{z^3}{3!} + 1\frac{z^4}{4!} + i\frac{z^5}{5!} + \dots \\ &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \dots + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots\right) = \cos(z) + i\sin(z). \end{aligned}$$

The last equality follows from recognizing the Taylor expansion of the cosine and the sine function.

Then we can find (1) and (2) as follows:

$$\begin{aligned} \frac{e^{iz} + e^{-iz}}{2} &= \frac{\cos(z) + i\sin(z) + \cos(-z) + i\sin(-z)}{2} \\ &= \frac{2\cos(z) + i\sin(z) - i\sin(z)}{2} \\ &= \cos(z), \end{aligned}$$

$$\begin{aligned} \frac{e^{iz} - e^{-iz}}{2i} &= \frac{\cos(z) + i\sin(z) - \cos(-z) - i\sin(-z)}{2i} \\ &= \frac{i\sin(z) + i\sin(z)}{2i} \\ &= \sin(z). \end{aligned}$$

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