MATH-207(d) Analysis IV
Exercise session 12

Exercice 1. Compute the Laplace transform of the function $f : [0, \infty) \to \mathbb{R}$ such that

$$f(t) = \begin{cases} 0 & \text{for } 0 \le t \le a \\ 1 & \text{for } t > a \end{cases}. \tag{1}$$

Answer. By definition, the Laplace transform of f is defined as

$$\mathfrak{L}[f](z) = \int_0^\infty f(t)e^{-tz}\,\mathrm{d}t = \int_a^\infty e^{-tz}\,\mathrm{d}t$$

for all $z \in \mathbb{C}$ such that the above integral converges. Studying now the convergence of this integral we deduce that if $\mathrm{Re}z > 0$ then

$$\int_a^\infty e^{-tz}\,\mathrm{d}t = -\frac{1}{z}\left(\lim_{t\to\infty} e^{-tz} - e^{-az}\right) = \frac{e^{-az}}{z}.$$

The same reasoning implies that if $\mathrm{Re}z \le 0$ then

$$\int_a^\infty e^{-zt}\,\mathrm{d}t$$

does not converge. Consequently, the Laplace transform of f, denoted $\mathfrak{L}[f](z)$ exists for all $z \in \mathbb{C}$ such that $\mathrm{Re}z > 0$, and it has the expression

$$\mathfrak{L}[f](z) = \frac{e^{-az}}{z}.$$

■

Exercice 2. Compute the Laplace transforms of the functions

$$f(t) = e^{at}\cosh(bt), \qquad g(t) = e^{at}\cosh(bt), \tag{2}$$

where $a, b \in \mathbb{C}$.

Answer. Recall that the complex hyperbolic cosine and complex hyperbolic sine functions can be expressed in terms of complex exponentials as

$$\cosh(bt) = \frac{1}{2}\left(e^{bt} + e^{-bt}\right) \quad \text{and} \quad \sinh(bt) = \frac{1}{2}\left(e^{bt} - e^{-bt}\right).$$

In order to compute the required Laplace transforms therefore, let us first introduce, for any fixed $\omega \in \mathbb{C}$, the function $h_\omega \colon \mathbb{R} \to \mathbb{C}$ as

$$h_\omega(t) = e^{t\omega}, \quad \forall t \in \mathbb{R}.$$

It follows that the Laplace transform of h_ω is given by

$$\mathfrak{L}[h_\omega](z) = \int_0^\infty h_\omega(t)e^{-tz}\,\mathrm{d}t = \int_0^\infty e^{t\omega}e^{-tz}\,\mathrm{d}t = \int_0^\infty e^{-t(z-\omega)}\,\mathrm{d}t,$$

for all $z \in \mathbb{C}$ such that the above integral converges. Studying now the convergence of this integral we deduce that if $\mathrm{Re}z > \mathrm{Re}\omega$ then

$$\int_0^\infty e^{-t(z-\omega)}\,\mathrm{d}t = -\frac{1}{z-\omega}\left(\lim_{t\to\infty} e^{-t(z-\omega)}\right) = \frac{1}{z-\omega}.$$

The same reasoning implies that if $\mathrm{Re}z \leq \mathrm{Re}\omega$ then

$$\int_0^\infty e^{-t(z-\omega)}\,\mathrm{d}t$$

does not converge. Consequently, the Laplace transform of h_ω, denoted $\mathfrak{L}[h_\omega](z)$ exists for all $z \in \mathbb{C}$ such that $\mathrm{Re}z > \mathrm{Re}\omega$, and it has the expression

$$\mathfrak{L}[h_\omega](z) = \frac{1}{z-\omega}. \tag{3}$$

Returning now to our original task, let us first consider the function f. By the definition of the Laplace transform and making use of Equation (3), we have that

$$\begin{aligned}
\mathfrak{L}[f](z) = \int_0^\infty f(t)e^{-tz}\,\mathrm{d}t &= \int_0^\infty e^{at}\cosh(bt)e^{-tz}\,\mathrm{d}t \\
&= \frac{1}{2}\int_0^\infty e^{at}e^{bt}e^{-tz}\,\mathrm{d}t + \frac{1}{2}\int_0^\infty e^{at}e^{-bt}e^{-tz}\,\mathrm{d}t \\
&= \frac{1}{2}\int_0^\infty e^{(a+b)t}e^{-tz}\,\mathrm{d}t + \frac{1}{2}\int_0^\infty e^{(a-b)t}e^{-tz}\,\mathrm{d}t \\
&= \frac{1}{2}\left(\mathfrak{L}[h_{a+b}](z) + \mathfrak{L}[h_{a-b}](z)\right) \\
&= \frac{1}{2}\left(\frac{1}{z-(a+b)} + \frac{1}{z-(a-b)}\right) \\
&= \frac{z-a}{(z-a)^2-b^2},
\end{aligned}$$

for all $z \in \mathbb{C}$ such that $\mathrm{Re}z > \max\{\mathrm{Re}(a+b), \mathrm{Re}(a-b)\}$. Next, let us consider the Laplace transform of the function g. In this case, making use of Equation (3), we have that

$$\begin{aligned}
\mathfrak{L}[g](z) = \int_0^\infty g(t)e^{-tz}\,\mathrm{d}t &= \int_0^\infty e^{at}\sinh(bt)e^{-tz}\,\mathrm{d}t \\
&= \frac{1}{2}\int_0^\infty e^{at}e^{bt}e^{-tz}\,\mathrm{d}t - \frac{1}{2}\int_0^\infty e^{at}e^{-bt}e^{-tz}\,\mathrm{d}t \\
&= \frac{1}{2}\int_0^\infty e^{(a+b)t}e^{-tz}\,\mathrm{d}t - \frac{1}{2}\int_0^\infty e^{(a-b)t}e^{-tz}\,\mathrm{d}t \\
&= \frac{1}{2}\left(\mathfrak{L}[h_{a+b}](z) - \mathfrak{L}[h_{a-b}](z)\right) \\
&= \frac{1}{2}\left(\frac{1}{z-(a+b)} - \frac{1}{z-(a-b)}\right) \\
&= \frac{b}{(z-a)^2-b^2},
\end{aligned}$$

for all $z \in \mathbb{C}$ such that $\mathrm{Re}z > \max\{\mathrm{Re}(a+b), \mathrm{Re}(a-b)\}$. ■

Exercice 3. Solve the ordinary differential equation

$$y'(t) + ay(t) = f(t), \quad t > 0, \tag{4}$$
$$y(0) = 1 \tag{5}$$

where $f(t) = te^{-3t}$. You are allowed to use all Laplace transforms that have been computed in the lecture or in previous exercises.

Answer. Applying the Laplace transform for both sides of the ordinary differential equation (ODE), and making use of the linearity of the Laplace transform as well as its derivative property, we obtain that

$$zY(z) - y(0) + aY(z) = \mathfrak{L}[f](z).$$

Here, Y denotes the Laplace transform of y. Using the initial condition, we further deduce that

$$Y(z) = \frac{1}{z+a}\mathfrak{L}[f](z) + \frac{1}{z+a}.$$

Recall that inverse Laplace transform of the function $G(z) = (z+a)^{-1}$ is given by (see, e.g., the previous exercise)

$$\mathfrak{L}^{-1}[G](t) = e^{-at} := g(t).$$

Consequently, the linearity of the Laplace transform combined with the convolution theorem for the Laplace transform implies that

$$\begin{aligned}
y(t) = \mathfrak{L}^{-1}[Y](t) &= g \star f + \mathfrak{L}^{-1}[G](t) \\
&= \int_0^t g(s)f(t-s)\,\mathrm{d}s + g(t) \\
&= \int_0^t e^{-as}(t-s)e^{-3(t-s)}\,\mathrm{d}s + e^{-at} \\
&= te^{-3t}\int_0^t e^{-as}e^{3s}\,\mathrm{d}s - e^{-3t}\int_0^t e^{-as}se^{3s}\,\mathrm{d}s + e^{-at}.
\end{aligned}$$

There are now two situations.
Case: $a = 3$**.** In this case, the above expression considerably simplies and we obtain

$$\begin{aligned}
y(t) = &= te^{-3t}\int_0^t 1\,\mathrm{d}s - e^{-3t}\int_0^t s\,\mathrm{d}s + e^{-at} \\
&= t^2e^{-3t} - \frac{1}{2}e^{-3t}t^2 + e^{-3t} \\
&= \frac{1}{2}e^{-3t}t^2 + e^{-3t}
\end{aligned}$$

Case: $a \neq 3$**.** In this case, we obtain, on the one hand, that

$$te^{-3t}\int_0^t e^{-as}e^{3s}\,\mathrm{d}s = \frac{te^{-3t}}{3-a}[e^{s(3-a)}]_{s=0}^{s=t} = \frac{te^{-3t}}{3-a}e^{t(3-a)} - \frac{te^{-3t}}{3-a} = \frac{te^{-at}}{3-a} - \frac{te^{-3t}}{3-a}.$$

On the other hand, the change of variables $r = s(3-a)$ and integration by parts yields that

$$\begin{aligned}
e^{-3t}\int_0^t e^{-as}se^{3s}\,\mathrm{d}s = e^{-3t}\int_0^t se^{s(3-a)}\,\mathrm{d}s &= \frac{1}{(3-a)^2}e^{-3t}\int_0^{t(3-a)} re^r\,\mathrm{d}r \\
&= \frac{1}{(3-a)^2}e^{-3t}\left(e^{t(3-a)}\left(t(3-a)-1\right)+1\right) \\
&= \frac{te^{-at}}{3-a} - \frac{e^{-at}}{(3-a)^2} + \frac{e^{-3t}}{(3-a)^2}.
\end{aligned}$$

Consequently, the solution to the ODE in the case $a \neq 3$ is given by

$$\begin{aligned} y(t) &= \frac{te^{-at}}{3-a} - \frac{te^{-3t}}{3-a} - \left(\frac{te^{-at}}{3-a} - \frac{e^{-at}}{(3-a)^2} + \frac{e^{-3t}}{(3-a)^2} \right) + e^{-at} \\ &= -\frac{te^{-3t}}{3-a} + \frac{e^{-at}}{(3-a)^2} - \frac{e^{-3t}}{(3-a)^2} + e^{-at}. \end{aligned}$$

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Exercice 4. Solve the ordinary differential equation

$$y''(t) + y(t) + 2y'(t) = f(t), \quad t > 0, \tag{6}$$
$$y(0) = 1, \quad y'(0) = 1. \tag{7}$$

in the two cases

(a) $f(t) = 0$,

(b) $f(t) = e^{-3t}$.

You are allowed to use all Laplace transforms that have been computed in the lecture or in previous exercises.

Answer. Applying the Laplace transform for both sides of the ordinary differential equation (ODE), and making use of the linearity of the Laplace transform as well as its derivative property, we obtain that

$$z^2Y(z) - zy(0) - y'(0) + Y(z) + 2zY(z) - 2y(0) = \mathfrak{L}[f](z).$$

Here, Y denotes the Laplace transform of y. Using the initial conditions, we further deduce that

$$Y(z) = \frac{1}{z^2+2z+1}\mathfrak{L}[f](z) + \frac{3+z}{z^2+2z+1}. \tag{8}$$

The goal now is to take the inverse Laplace transform on both sides of Equation (8). In order to prepare for this, let us first notice that we can write

$$\frac{3+z}{z^2+2z+1} = \frac{1+z}{z^2+2z+1} + \frac{2}{z^2+2z+1} = \frac{1}{z+1} + \frac{2}{z^2+2z+1}.$$

Let us therefore introduce the functions $G(z) = (z+1)^{-1}$ and $H(z) = (z^2+2z+1)^{-1}$. Recalling the Laplace transforms that have been computed in previous exercises (see, in particular Exercise 2 above and Exercise 3 from Sheet 11) , we deduce that

$$\mathfrak{L}^{-1}[G](t) = e^{-t} := g(t), \quad \text{and} \quad \mathfrak{L}^{-1}[H](t) = te^{-t} := h(t).$$

Returning now to Equation (8), we make use of the linearity of the Laplace transform together with the convolution theorem to deduce that

$$\begin{aligned} y(t) = \mathfrak{L}^{-1}[Y](t) &= h \star f + \mathfrak{L}^{-1}[G](t) + 2\mathfrak{L}^{-1}[H](t) \\ &= \int_0^t h(s)f(t-s)\, \mathrm{d}s + g(t) + 2h(t) \\ &= \int_0^t h(s)f(t-s)\, \mathrm{d}s + e^{-t} + 2te^{-t}. \end{aligned}$$

It remains to compute the convolution. Once again, there are two cases:
Case: $f = 0$. In this case, the convolution is just zero and we obtain that

$$y(t) = e^{-t} + 2te^{-t}.$$

Case: $f(t) = e^{-3t}$. In this case, the convolution can be written as

$$\begin{aligned}\int_0^t h(s)f(t-s)\,\mathrm{d}s = \int_0^t se^{-s}e^{-3(t-s)}\,\mathrm{d}s = \int_0^t se^{-s}e^{-3(t-s)}\,\mathrm{d}s &= e^{-3t}\int_0^t se^{2s}\,\mathrm{d}s \\ &= \frac{1}{4}e^{-3t}\int_0^{2t} re^r\,\mathrm{d}r,\end{aligned}$$

where the last step uses the change of variables $r = 2s$. Using now integration by parts yields

$$\begin{aligned}\int_0^t h(s)f(t-s)\,\mathrm{d}s = \frac{1}{4}e^{-3t}\int_0^{2t} re^r\,\mathrm{d}r &= \frac{1}{4}e^{-3t}\left(e^{2t}(2t-1)+1\right) \\ &= \frac{1}{4}e^{-t}(2t-1) + \frac{1}{4}e^{-3t}.\end{aligned}$$

Consequently, the solution to the ODE in the case $f(t) = e^{-3t}$ is given by

$$\begin{aligned}y(t) &= \frac{1}{4}e^{-t}(2t-1) + \frac{1}{4}e^{-3t} + e^{-2t} + 2te^{-t} \\ &= \frac{10}{4}te^{-t} + \frac{3}{4}e^{-t} + \frac{1}{4}e^{-3t}.\end{aligned}$$

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Exercice 5. Find a solution to the ordinary differential equation

$$y'''(t) + y'(t) = te^{-t}, \tag{9}$$

$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1 \tag{10}$$

via the Laplace transform. You are allowed to use all Laplace transforms that have been computed in the lecture or in previous exercises.

Hint: The following identity might be useful:

$$\frac{1}{z^3+z} = \frac{1}{z} - \frac{z}{z^2+1}.$$

Answer. Applying the Laplace transform for both sides of the ordinary differential equation (ODE), and making use of the linearity of the Laplace transform as well as its derivative property, we obtain that

$$z^3Y(z) - z^2y(0) - zy'(0) - y''(0) + zY(z) - y(0) = \mathfrak{L}[f](z).$$

Here, Y denotes the Laplace transform of y and we have introduced the function $f(t) = te^{-t}$. Using the initial conditions, we further deduce that

$$Y(z) = \frac{1}{z^3+z}\mathfrak{L}[f](z) + \frac{1}{z^3+z}. \tag{11}$$

The goal now is to take the inverse Laplace transform on both sides of Equation (11). In order to prepare for this, we require the inverse Laplace transform of the function $G(z) = (z^3+z)^{-1}$.

To this end, let us first notice the Laplace transform of the cosine function is given by

$$\mathfrak{L}[\cos](z) = \frac{z}{z^2+1}.$$

Moreover, the Laplace transform of 1 is given by

$$\mathfrak{L}[1](z) = \frac{1}{z}.$$

Since

$$\frac{1}{z} - \frac{z}{z^2+1} = \frac{1}{z^3+z},$$

we deduce from the linear property of the Laplace transform that the inverse Laplace transform of G is given by

$$g(t) := \mathfrak{L}^{-1}[G](t) = \mathfrak{L}^{-1}\big[\mathfrak{L}[1]\big](t) - \mathfrak{L}^{-1}\big[\mathfrak{L}[\cos]\big](t) = 1 - \cos(t).$$

Returning now to Equation (11), we make use of the linearity of the Laplace transform together with the convolution theorem to deduce that

$$\begin{aligned} y(t) = \mathfrak{L}^{-1}[Y](t) = (g \star f)(t) + g(t) \\ &= \int_0^t g(s) f(t-s) \, \mathrm{d}s + g(t) \\ &= \int_0^t \big(1-\cos(s)\big)(t-s)e^{-(t-s)} \, \mathrm{d}s + 1 - \cos(t) \\ &= \int_0^t (t-s)e^{-(t-s)} \, \mathrm{d}s - \int_0^t \cos(s)(t-s)e^{-(t-s)} \, \mathrm{d}s + 1 - \cos(t). \end{aligned} \tag{12}$$

The first integral above can be computed using a change of variables $w = t - s$ followed by integration by parts. This yields

$$\int_0^t (t-s)e^{-(t-s)} \, \mathrm{d}s = -\int_t^0 we^{-w} \, \mathrm{d}w = \int_0^t we^{-w} \, \mathrm{d}w = 1 - e^{-t}(t+1).$$

The second integral can be computed with similar arguments if we write the cos function in terms of exponentials. We find

$$\begin{aligned} &\int_0^t \cos(s)(t-s)e^{-(t-s)} \, \mathrm{d}s \\ &= te^{-t} \int_0^t \cos(s)e^s \, \mathrm{d}s - e^{-t} \int_0^t \cos(s)se^s \, \mathrm{d}s \\ &= te^{-t} \int_0^t \frac{e^{is}+e^{-is}}{2} e^s \, \mathrm{d}s - e^{-t} \int_0^t \frac{e^{is}+e^{-is}}{2} se^s \, \mathrm{d}s \\ &= te^{-t} \left(\frac{1}{2} \int_0^t e^{s+is} \, \mathrm{d}s + \frac{1}{2} \int_0^t e^{s-is} \, \mathrm{d}s \right) - e^{-t} \left(\frac{1}{2} \int_0^t se^{s+is} \, \mathrm{d}s + \frac{1}{2} \int_0^t se^{s-is} \, \mathrm{d}s \right). \end{aligned}$$

We need to compute four integrals. The first two of the remaining integrals are standard:

$$\int_0^t e^{s+is} \, \mathrm{d}s = \frac{e^{t+it}-1}{1+i}, \qquad \int_0^t e^{s-is} \, \mathrm{d}s = \frac{e^{t-it}-1}{1-i}.$$

This leads us to

$$\begin{aligned}\frac{1}{2}\int_0^t e^{s+is}\,\mathrm{d}s+\frac{1}{2}\int_0^t e^{s-is}\,\mathrm{d}s &= \frac{(1-i)e^{t+it}-1+i}{4}+\frac{(1+i)e^{t-it}-1-i}{4}\\ &= \frac{e^{t+it}+e^{t-it}-ie^{t+it}+ie^{t-it}}{4}-\frac{1}{2}\\ &= e^t\frac{e^{it}+e^{-it}}{4}+e^t\frac{e^{it}-e^{-it}}{4i}-\frac{1}{2}\\ &= e^t\frac{\cos(t)}{2}+e^t\frac{\sin(t)}{2}-\frac{1}{2}.\end{aligned}$$

The other two integrals can be computed similar as above, or using integral by parts:

$$\begin{aligned}\int_0^t se^{s(1+i)}\,\mathrm{d}s &= \frac{1}{1+i}[se^{s(1+i)}]_{s=0}^{s=t}-\frac{1}{1+i}\int_0^t e^{s(1+i)}\,\mathrm{d}s\\ &= \frac{1}{1+i}[se^{s(1+i)}]_{s=0}^{s=t}-\frac{1}{(1+i)^2}[e^{s(1+i)}]_{s=0}^{s=t}\\ &= \frac{1}{1+i}te^{t(1+i)}-\frac{1}{(1+i)^2}e^{t(1+i)}.\end{aligned}$$

A similar calculation yields

$$\begin{aligned}\int_0^t se^{s(1-i)}\,\mathrm{d}s &= \frac{1}{1-i}[se^{s(1-i)}]_{s=0}^{s=t}-\frac{1}{1-i}\int_0^t e^{s(1-i)}\,\mathrm{d}s\\ &= \frac{1}{1-i}[se^{s(1-i)}]_{s=0}^{s=t}-\frac{1}{(1-i)^2}[e^{s(1-i)}]_{s=0}^{s=t}\\ &= \frac{1}{1-i}te^{t(1-i)}-\frac{1}{(1-i)^2}e^{t(1-i)},\end{aligned}$$

This leads to

$$\begin{aligned}&\int_0^t se^{s-is}\,\mathrm{d}s+\int_0^t se^{s-is}\,\mathrm{d}s\\ &= \frac{1}{1+i}te^{t(1+i)}-\frac{1}{(1+i)^2}e^{t(1+i)}+\frac{1}{1-i}te^{t(1-i)}-\frac{1}{(1-i)^2}e^{t(1-i)}\\ &= \frac{1-i}{2}te^{t(1+i)}-\frac{1-2i-1}{4}e^{t(1+i)}+\frac{1+i}{2}te^{t(1-i)}-\frac{1+2i-1}{4}e^{t(1-i)}\\ &= t\frac{e^{t(1+i)}+e^{t(1-i)}}{2}+t\frac{e^{t(1+i)}-e^{t(1-i)}}{2i}-\frac{e^{t(1+i)}-e^{t(1-i)}}{2i}\\ &= e^t t\cos(t)+e^t t\sin(t)-e^t\sin(t).\end{aligned}$$

Putting these together, we get

$$\begin{aligned}&\int_0^t \cos(s)(t-s)e^{-(t-s)}\,\mathrm{d}s\\ &= te^{-t}\left(e^t\frac{\cos(t)}{2}+e^t\frac{\sin(t)}{2}-\frac{1}{2}\right)-e^{-t}\frac{1}{2}\left(e^t t\cos(t)+e^t t\sin(t)-e^t\sin(t)\right)\\ &= \frac{\sin(t)}{2}-\frac{t}{2}e^{-t}.\end{aligned}$$

Returning now to Equation (12), we obtain that

$$\begin{aligned}
y(t) &= \int_0^t (t-s)e^{-(t-s)}\,\mathrm{d}s - \int_0^t \cos(s)(t-s)e^{-(t-s)}\,\mathrm{d}s + 1 - \cos(t) \\
&= 1 - e^{-t}(t+1) - \frac{\sin(t)}{2} + \frac{t}{2}e^{-t} + 1 - \cos(t) \\
&= 2 - e^{-t}(\frac{t}{2}+1) - \frac{\sin(t)}{2} - \cos(t).
\end{aligned}$$

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Exercice 6. Solve the initial value problem

$$\begin{aligned}
u''''(t) &= e^{-t}, \quad t>0, \\
u(0) = u'(0) = u''(0) &= u'''(0) = 0.
\end{aligned}$$

Answer. Applying the Laplace transform for both sides of the ordinary differential equation (ODE), and making use of the derivative property, we obtain that

$$z^4U(z) - z^3u(0) - z^2u'(0) - zu''(0) - u'''(0) = \frac{1}{1+z}.$$

Here, we denote by U the Laplace transform of the solution function u, and we have used the fact that the Laplace transform of the function $f(t) = e^{-t}$ is given by $\mathfrak{L}[f](z) = (1+z)^{-1}$. Applying now the initial conditions yields the algebraic equation

$$U(z) = \frac{1}{z^4(1+z)}. \tag{13}$$

Only possible solution involves the **convolution formula**. We know that z^{-4} is the Laplace transform of $t^3/6$. By the convolution formula,

$$u(t) = \frac{1}{6}\int_0^t s^3e^{-(t-s)}ds = \frac{e^{-t}}{6}\int_0^t s^3e^s ds.$$

We use integration by parts several times:

$$\begin{aligned}
u(t) &= \int_0^t s^3e^s ds \\
&= t^3e^t - 3\int_0^t s^2e^s ds \\
&= t^3e^t - 3t^2e^t + 6\int_0^t se^s ds \\
&= t^3e^t - 3t^2e^t + 6te^t - \int_0^t e^s ds \\
&= t^3e^t - 3t^2e^t + 6te^t - 6e^t + 6.
\end{aligned}$$

Putting this together, we find the solution

$$\begin{aligned}
u(t) &= \frac{e^{-t}}{6}\left(t^3e^t - 3t^2e^t + 6te^t - 6e^t + 6\right) \\
&= \frac{t^3}{6} - \frac{t^2}{2} + t - 1 + e^{-t}.
\end{aligned}$$

Alternatively, we can use the **partial fraction decomposition**. Before proceeding with the application of the inverse Laplace transform let us first compute the partial fraction decomposition of the term on the right. We begin with the ansatz

$$\frac{1}{z^4(1+z)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z^3} + \frac{D}{z^4} + \frac{E}{1+z},$$

which then yields the equation

$$1 = Az^3(1+z) + Bz^2(1+z) + Cz(1+z) + D(1+z) + Ez^4.$$

Substituting $z = -1$ immediately yields $E = 1$ while substituting $z = 0$ yields $D = 1$. To obtain the remaining constants A, B, C, we can simply substitute $z = 1, \pm 2$ to obtain the system of equations

$$\begin{aligned} -2 &= 2A + 2B + 2C \\ -18 &= 24A + 12B + 6C \\ -14 &= 8A - 4B + 2C. \end{aligned}$$

Combining the second and third equation we easily deduce that

$$-18 + 42 = 24B \qquad \Longrightarrow \; B = 1.$$

Further more, combining the first and thiird equation yields

$$12 = -6A + 6B \qquad \Longrightarrow \; A = -1.$$

Consequently, we obtain that $C = -1$. We thus conclude that

$$U(z) = \frac{1}{z^4(1+z)} = -\frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \frac{1}{z^4} + \frac{1}{1+z}.$$

Applying the inverse Laplace transform and making use of it's linearity, we deduce that

$$u(t) = -1 + t - \frac{t^2}{2} + \frac{t^3}{6} + e^{-t}.$$

This is the solution to the differential equation. ■

Exercice 7. Let $z^2 + pz + q$ be a polynomial with real coefficients $p, q \in \mathbb{R}$ and roots $z_1, z_2 \in \mathbb{C}$. Use the assumption that $p, q \in \mathbb{R}$ to show the following:

- The imaginary parts of the roots satisfy $\Im z_1 = -\Im z_2$.
- If any of the roots is not real, then both roots are not real, and then their real parts satisfy $\Re z_1 = \Re z_2$.

Answer. We write $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Since $z_1, z_2 \in \mathbb{C}$ are the roots of the polynomial, we have

$$z^2 + pz + q = (z - z_1)(z - z_2) = z^2 + (z_1 + z_2)z + z_1 z_2.$$

- Since $p \in \mathbb{R}$, we have $z_1 + z_2 \in \mathbb{R}$, and so we conclude $y_1 = -y_2$. That shows the first assertion.

- We obviously have
$$q = z_1 z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1) i.$$
Since $q \in \mathbb{R}$, we see that
$$0 = x_1 y_2 + x_2 y_1 = x_1 y_2 - x_2 y_2 = y_2 (x_1 - x_2).$$
If one of the roots is not real, then this means that $y_1 \neq 0$ or $y_2 \neq 0$. But then $y_1 = -y_2$ implies that both $y_1 \neq 0$ and $y_2 \neq 0$. Hence $0 = y_2(x_1 - x_2)$ can only hold if $x_1 = x_2$.

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Exercice 8. Let $p, q \in \mathbb{R}$. Find the Laplace transform of any solution of
$$y'' + py' + qy = f.$$
Use this approach to find the solution in the special case $f = 0$ with initial conditions $y(0) = 0$ and $y'(0) = 2$ and with parameters $p = 0$ and $q = 1$.

Answer. Applying the Laplace transform to the differential equation, we find
$$z^2 Y(z) - y(0) z - y'(0) + pzY(z) - py(0) + qY(z) = F(z).$$
Isolating $Y(z)$ yields
$$\left(z^2 + pz + q\right) Y(z) = F(z) + y(0) z + y'(0) + py(0)$$
and so
$$Y(z) = \frac{F(z) + y(0) z + y'(0) + py(0)}{z^2 + pz + q}$$
In the special case $f = 0$ with $y(0) = 0$ and $y'(0) = 2$ and with $p = 0$ and $q = 1$, we thus have
$$Y(z) = \frac{2}{z^2 + pz + q} = \frac{2}{z^2 + 1} = \frac{2}{(z+i)(z-i)}.$$
We check that $1/(z+i)$ and $1/(z-i)$ are the Laplace transforms of e^{-it} and e^{it}, respectively. Via the convolution formula,
$$y(t) = 2 \int_0^t e^{is} e^{-i(t-s)} = 2e^{-it} \int_0^t e^{2is} = 2e^{-it} \frac{1}{2i} \left(e^{2it} - 1\right) = 2 \frac{e^{it} - e^{-it}}{2i} = 2 \sin(t),$$
which is the solution. *Remark: more generally, $z^2 + pz + q = (z - z_1)(z - z_2)$ has two roots $z_1, z_2 \in \mathbb{C}$. The solution can always be found using the approach above, via the convolution formula.* ■

Exercice 9 (Extra)**.** The n-moment of a function $f : [0, \infty) \to \mathbb{R}$ is defined as
$$\mu_n := \int_0^\infty t^n f(t) \, \mathrm{d}t, \tag{14}$$
provided that this integral converges. Show that if all n-th moments of f converge and
$$\sup_{n \in \mathbb{N}} \int_0^\infty t^n |f(t)| \, \mathrm{d}t = \nu < \infty,$$
then
$$\mathfrak{L}[f](z) = \sum_{n=0}^\infty \frac{(-1)^n}{n!} \mu_n z^n. \tag{15}$$

Answer. On the one hand, recall that the Laurent series expansion at $z_0 = 0$ for the complex exponential takes the form of an infinite sum

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \tag{16}$$

which is valid for all $z \in \mathbb{C}$. On the other hand, by definition of the Laplace transform,

$$\mathfrak{L}[f](z) = \int_0^{\infty} f(t)e^{-tz} \, \mathrm{d}t,$$

for all $z \in \mathbb{C}$ such that the above integral converges. Plugging in Equation (16) in the expression for the above Laplace transform, we deduce that

$$\mathfrak{L}[f](z) = \int_0^{\infty} f(t) \sum_{n=0}^{\infty} (-1)^n \frac{t^n z^n}{n!} \, \mathrm{d}t = \int_0^{\infty} \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!} f(t) t^n \, \mathrm{d}t.$$

We would now like to interchange the infinite summation and the integral. Indeed, if can do that, then

$$\sum_{n=0}^{\infty} \int_0^{\infty} (-1)^n \frac{z^n}{n!} f(t) t^n \, \mathrm{d}t = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!} \int_0^{\infty} f(t) t^n \, \mathrm{d}t = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!} \mu_n.$$

However, it is not immediately obvious that we can exchange the integral and the infintie sum because the linearity of the integral a priori only applies to a *finite* sum.
We technically must justify the exchange of sum and integral. To simplify the notation, we introduce the functions $\{g_n\}_{n \in \mathbb{N}}$ defined as

$$g_n(t) = (-1)^n \frac{z^n}{n!} f(t) t^n.$$

We want to show that

$$\sum_{n=0}^{\infty} \int_0^{\infty} g_n(t) \, \mathrm{d}t = \int_0^{\infty} \sum_{n=0}^{\infty} g_n(t) \, \mathrm{d}t.$$

This equation follows from Fubini's theorem if we can show that the infinite sum and integral is absolutely convergent. The latter is seen from

$$\sum_{n=0}^{\infty} \int_0^{\infty} |g_n(t)| \, \mathrm{d}t = \sum_{n=0}^{\infty} \frac{|z|^n}{n!} \int_0^{\infty} t^n \, |f(t)| \, \mathrm{d}t \leq \sum_{n=0}^{\infty} \frac{|z|^n}{n!} \nu = \nu \cdot \exp(|z|).$$

This justifies why we can exchange limit and integral. ■