MATH-207(d) Analysis IV
Exercise session 11

Exercice 1. Let $a, b, c \in \mathbb{R}$ with $c \neq 0$. Given the Fourier transform $\hat{f}$ of f, you are asked to find the Fourier transform of:

- $g(x) = f(x+a)$
- $g(x) = e^{-ibx} f(x+a)$
- $g(x) = f(x/c)$ with $c \neq 0$.

Answer. Note that these Fourier transforms can be easily obtained by using the properties of the Fourier transform discussed in the lecture. For the sake of completness, we state the direct calculations below.

- By definition, the Fourier transform of g is given by

$$\widehat{g}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-i\alpha x} \, \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x+a) e^{-i\alpha x} \, \mathrm{d}x.$$

  We use the change of variables $y = x + a$ to obtain

$$\begin{aligned} \widehat{g}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x+a) e^{-i\alpha x} \, \mathrm{d}x &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-i\alpha(y-a)} \, \mathrm{d}y \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-i\alpha y} e^{i\alpha a} \, \mathrm{d}y \\ &= e^{i\alpha a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-i\alpha y} \, \mathrm{d}y \\ &= e^{i\alpha a} \widehat{f}(\alpha). \end{aligned}$$

- By definition, the Fourier transform of g is given by

$$\begin{aligned} \widehat{g}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-i\alpha x} \, \mathrm{d}x &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ibx} f(x+a) e^{-i\alpha x} \, \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x+a) e^{-i(\alpha+b)x} \, \mathrm{d}x. \end{aligned}$$

  Once again, we use the change of variables $y = x + a$ to obtain

$$\begin{aligned} \widehat{g}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x+a) e^{-i(\alpha+b)x} \, \mathrm{d}x &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-i(\alpha+b)(y-a)} \, \mathrm{d}y \\ &= e^{i(\alpha+b)a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-i(\alpha+b)y} \, \mathrm{d}y \\ &= e^{i(\alpha+b)a} \widehat{f}(\alpha+b). \end{aligned}$$

- By definition, the Fourier transform of g is given by

$$\widehat{g}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-i\alpha x} \, \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x/c) e^{-i\alpha x} \, \mathrm{d}x.$$

In this case, we will use the change of variables $y = x/c$. There are now two cases.

Case $c > 0$:

In this case, note that $\lim x \to \infty \implies \lim y \to \infty$ and $\lim x \to -\infty \implies \lim y \to -\infty$. We thus obtain

$$\begin{aligned}\widehat{g}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x/c) e^{-i\alpha x} \, \mathrm{d}x &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c f(y) e^{-i\alpha c y} \, \mathrm{d}y \\ &= c \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-i(\alpha c) y} \, \mathrm{d}y \qquad = c\widehat{f}(c\alpha).\end{aligned}$$

Case $c < 0$:

In this case, note that $\lim x \to \infty \implies \lim y \to -\infty$ and $\lim x \to -\infty \implies \lim y \to \infty$. We thus obtain

$$\begin{aligned}\widehat{g}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x/c) e^{-i\alpha x} \, \mathrm{d}x &= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} c f(y) e^{-i\alpha c y} \, \mathrm{d}y \\ &= -c \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-i(\alpha c) y} \, \mathrm{d}y \qquad = -c\widehat{f}(c\alpha).\end{aligned}$$

Combining the above two cases, we conclude that

$$\widehat{g}(\alpha) = |c| \widehat{f}(c\alpha).$$

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Exercice 2. Compute the Laplace transforms of the following functions

$$f(t) = e^{i\theta t}, \quad g(t) = \sin(\omega t), \quad \text{and} \quad h(t) = \cos(\omega t),$$

where $\theta \in \mathbb{R}$ and $\omega > 0$.

Answer. By definition, the Laplace transform of f is defined as

$$\mathfrak{L}[f](z) = \int_0^{\infty} f(t) e^{-tz} \, \mathrm{d}t = \int_0^{\infty} e^{i\theta t} e^{-tz} \, \mathrm{d}t,$$

for all $z \in \mathbb{C}$ such that the above integral converges. Studying now the convergence of this integral we deduce that if $\mathrm{Re} z > 0$ then

$$\begin{aligned}\int_0^{\infty} e^{i\theta t} e^{-tz} \, \mathrm{d}t &= \int_0^{\infty} e^{-t(z - i\theta)} \, \mathrm{d}t \\ &= \frac{1}{-(z - i\theta)} \left(\lim_{L \to \infty} e^{-L(z - i\theta)} - e^{-0 \cdot (z - i\theta)} \right) = \frac{1}{(z - i\theta)},\end{aligned}$$

The same reasoning implies that if $\mathrm{Re} z \leq 0$ then

$$\int_0^{\infty} e^{i\theta t} e^{-tz} \, \mathrm{d}t$$

does not converge. Consequently, the Laplace transform of f, denoted $\mathfrak{L}[f](z)$ exists for all $z \in \mathbb{C}$ such that Re$z > 0$, and it has the expression

$$\mathfrak{L}[f](z) = \frac{1}{(z - i\theta)}.$$

Turning now to the function g, we see that the Laplace transform of g is defined as

$$\mathfrak{L}[g](z) = \int_0^\infty g(t)e^{-tz}\, \mathrm{d}t = \int_0^\infty \sin(\omega t)e^{-tz}\, \mathrm{d}t,$$

for all $z \in \mathbb{C}$ such that the above integral converges. Using the fact that

$$\sin(\omega t) = (e^{i\omega t} - e^{-i\omega t})/(2i)$$

we deduce that we can further write

$$\begin{aligned}\int_0^\infty \sin(\omega t)e^{-tz}\, \mathrm{d}t &= \frac{1}{2i}\int_0^\infty \left(e^{i\omega t} - e^{-i\omega t}\right) e^{-tz}\, \mathrm{d}t \\ &= \frac{1}{2i}\Big(\underbrace{\int_0^\infty e^{-t(z-i\omega)}\, \mathrm{d}t}_{:=\text{(I)}} - \underbrace{\int_0^\infty e^{-t(z+i\omega)}\, \mathrm{d}t}_{:=\text{(II)}}\Big).\end{aligned}$$

Notice now that the integral (I) is nothing else than the Laplace transform of the function f calculated above. Thus, we obtain that the integral (I) converges if and only if Re$z > 0$ in which case

$$\text{(I)} = \int_0^\infty e^{-t(z-i\omega)}\, \mathrm{d}t = \frac{1}{(z - i\omega)}.$$

The same calculation also yields that the integral (II) converges if and only if Re$z > 0$ in which case

$$\text{(II)} = \int_0^\infty e^{-t(z+i\omega)}\, \mathrm{d}t = \frac{1}{(z + i\omega)}.$$

Combining these results, we obtain that the Laplace transform of g, denoted $\mathfrak{L}[g](z)$ exists for all $z \in \mathbb{C}$ such that Re$z > 0$, and it has the expression

$$\mathfrak{L}[g](z) = \frac{1}{2i}\left(\frac{1}{(z - i\omega)} - \frac{1}{(z + i\omega)}\right) = \frac{\omega}{z^2 + \omega^2}.$$

Finally, we turn to the function h. The calculation of the Laplace transform of h is, in fact, very close to the corresponding calculation that we have done for the function g. Indeed, we see that the Laplace transform of h is given by

$$\mathfrak{L}[h](z) = \int_0^\infty g(t)e^{-tz}\, \mathrm{d}t = \int_0^\infty \cos(\omega t)e^{-tz}\, \mathrm{d}t,$$

for all $z \in \mathbb{C}$ such that the above integral converges. Using the fact that $\cos(\omega t) = (e^{i\omega t} + e^{-i\omega t})/(2)$ we deduce that we can further write

$$\begin{aligned}\int_0^\infty \sin(\omega t)e^{-tz}\, \mathrm{d}t &= \frac{1}{2}\int_0^\infty \left(e^{i\omega t} + e^{-i\omega t}\right) e^{-tz}\, \mathrm{d}t \\ &= \frac{1}{2}\Big(\underbrace{\int_0^\infty e^{-t(z-i\omega)}\, \mathrm{d}t}_{:=\text{(I)}} + \underbrace{\int_0^\infty e^{-t(z+i\omega)}\, \mathrm{d}t}_{:=\text{(II)}}\Big).\end{aligned}$$

Note that we have already computed the integrals (I) and (II) during our study of the function g. Making use of these expressions, we see that the integral (I) and (II) converge if and only if $\mathrm{Re}z > 0$ in which case

$$(\mathrm{I}) = \int_0^\infty e^{-t(z-i\omega)}\,\mathrm{d}t = \frac{1}{(z-i\omega)},$$
$$(\mathrm{II}) = \int_0^\infty e^{-t(z+i\omega)}\,\mathrm{d}t = \frac{1}{(z+i\omega)}.$$

We thus obtain that the Laplace transform of h, denoted $\mathfrak{L}[h](z)$ exists for all $z \in \mathbb{C}$ such that $\mathrm{Re}z > 0$, and it has the expression

$$\mathfrak{L}[h](z) = \frac{1}{2}\left(\frac{1}{(z-i\omega)} + \frac{1}{(z+i\omega)}\right) = \frac{z}{z^2+\omega^2}.$$

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Exercice 3. Compute the Laplace transform of

$$g(t) = te^{-at}. \tag{1}$$

Answer. We know that for any function $f(t)$ with Laplace transform $F(z)$, the complex derivative $F'(z)$ is the negative Laplace transform of $tf(t)$. Recognizing this, we pick $f(t) = e^{-at}$. Its Laplace transform

$$F(z) = \frac{1}{z+a} \tag{2}$$

has derivative

$$F'(z) = -\frac{1}{(z+a)^2}. \tag{3}$$

Consequently, the Laplace transform of $g(t) = tf(t)$ is

$$G(z) = -F'(z) = \frac{1}{(z+a)^2}. \tag{4}$$

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Exercice 4. Compute the Laplace transforms of

$$f_1(t) = e^{-\beta t}\cos(\gamma t), \quad f_2(t) = e^{-\beta t}\sin(\gamma t). \tag{5}$$

In the case $\beta = 0$, you the functions from the previous exercise.

Answer. The first step is to rewrite these functions via exponentials:

$$f_1(t) = e^{-\beta t}\frac{e^{i\gamma t} + e^{-i\gamma t}}{2} = \frac{e^{(-\beta+i\gamma)t} + e^{(-\beta-i\gamma)t}}{2}, \tag{6}$$
$$f_2(t) = e^{-\beta t}\frac{e^{i\gamma t} - e^{-i\gamma t}}{2i} = \frac{e^{(-\beta+i\gamma)t} - e^{(-\beta-i\gamma)t}}{2i}, \tag{7}$$

By linearity, the Laplace transforms are

$$
\begin{aligned}
F_1(z) &= \frac{1}{2}\mathfrak{L}\left[e^{(-\beta+i\gamma)t}\right] + \frac{1}{2}\mathfrak{L}\left[e^{(-\beta-i\gamma)t}\right] && (8)\\
&= \frac{1}{2(z+\beta-i\gamma)} + \frac{1}{2(z+\beta+i\gamma)} && (9)\\
&= \frac{1}{2(z+\beta-i\gamma)} + \frac{1}{2(z+\beta+i\gamma)} && (10)\\
&= \frac{z+\beta-i\gamma+z+\beta+i\gamma}{2(z+\beta)^2+2\gamma^2} && (11)\\
&= \frac{z+\beta+z+\beta}{2(z+\beta)^2+2\gamma^2} && (12)\\
&= \frac{z+\beta}{(z+\beta)^2+\gamma^2}. && (13)
\end{aligned}
$$

and

$$
\begin{aligned}
F_2(z) &= \frac{1}{2i}\mathfrak{L}\left[e^{(-\beta+i\gamma)t}\right] - \frac{1}{2i}\mathfrak{L}\left[e^{(-\beta-i\gamma)t}\right] && (14)\\
&= \frac{1}{2i(z+\beta-i\gamma)} - \frac{1}{2i(z+\beta+i\gamma)} && (15)\\
&= \frac{z+\beta+i\gamma-z-\beta+i\gamma}{2i(z+\beta)^2+2i\gamma^2} && (16)\\
&= \frac{\gamma}{(z+\beta)^2+\gamma^2}. && (17)
\end{aligned}
$$

These result to be shown. ■

Exercice 5. (a) Consider the functions $f, g : [0, \infty) \to \mathbb{C}$ with

$$f(t) = t, \quad g(t) = e^{-t}. \tag{18}$$

Compute the convolution $f \star g$. Determine its Laplace transform using the convolution formula, and verify the result using a diirect calculation.

Answer.

(a) Recall the Laplace transforms

$$\mathfrak{L}[t] = \frac{1}{z^2}, \qquad \mathfrak{L}[e^{-t}] = \frac{1}{z+1}.$$

By the convolution formula

$$\mathfrak{L}\left[t \star e^{-t}\right] = \frac{1}{z^2(z+1)}.$$

We have

$$
\begin{aligned}
(f \star g)(t) &= \int_0^t s e^{-(t-s)}\, ds = e^{-t}\int_0^t s e^s\, ds \\
&= e^{-t}\left[se^s\right]_{s=0}^{s=t} - e^{-t}\int_0^t e^s\, ds \\
&= e^{-t}\left[se^s\right]_{s=0}^{s=t} - e^{-t}\left[e^s\right]_{s=0}^{s=t} \\
&= e^{-t}\left(te^t\right) - e^t\left(e^t - 1\right) \\
&= t - 1 + e^{-t}.
\end{aligned}
$$

Applying the Laplace transform to each term, we obtain the function

$$\frac{1}{z^2} - \frac{1}{z} + \frac{1}{z+1} = \frac{1}{z^2(z+1)}. \tag{19}$$

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Exercice 6. Given $f : [0,1] \to \mathbb{R}$. Let $y : [0,1] \to \mathbb{R}$ such that

$$\begin{aligned} &y''(t) = f(t), \quad 0 < t < 1 \\ &y(0) = 0, \quad y(1) = 0 \end{aligned}$$

Determine if the following statements are true or false. Justify your answer.

(a) Let $F(z)$ and $Y(z)$ denote the Laplce transforms of f and y. We have

$$Y(z) = \frac{F(z)}{z^2} + \frac{y'(0)}{z^2}.$$

(b) $y(t) = \int_0^t f(s)(t-s)\mathrm{d}s + ty'(0)$.

(c) $y(t) = \int_0^t f(s)(t-s)\mathrm{d}s + t\int_0^1 f(s)(1-s)\mathrm{d}s$

Answer.

(a) TRUE. Apply Laplace transform to the equation to obtain

$$z^2Y(z) - y'(0) - zy(0) = F(z).$$

Rearranging and using the first boundary conditions gives

$$Y(z) = \frac{F(z)}{z^2} + \frac{y'(0)}{z^2}.$$

(b) TRUE. Recall $\mathcal{L}(t)(z) = \frac{1}{z^2}$. Using the property characterizing the Laplace transform of a convolution $\mathcal{L}(f * g)(z) = \mathcal{L}(f)(z)\mathcal{L}(g)(z)$ we get

$$y(t) = f(t) * t + ty'(0) = \int_0^t f(s)(t-s)\mathrm{d}s + ty'(0).$$

(c) FALSE. We use answer (b) and plug in the second boundary condition to find.

$$0 = y(1) = \int_0^1 f(s)(1-s)ds + y'(0) \implies y'(0) = -\int_0^1 f(s)(1-s)ds.$$

Therefore, the sign was wrong. The correct answer is

$$y(t) = \int_0^t f(s)(t-s)\mathrm{d}s - t\int_0^1 f(s)(1-s)\mathrm{d}s.$$

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Exercice 7. Consider the following system of differential equations

$$\begin{cases} x'(t) = 2x(t) - 3y(t), & t > 0 \\ y'(t) = y(t) - 2x(t), & t > 0 \\ x(0) = 8, y(0) = 3. \end{cases}$$

and denote $X(z) = \mathcal{L}(x)(z)$ and $Y(z) = \mathcal{L}(y)(z)$.

(a) Applying the Laplace transform to the equations we find that

$$\begin{aligned} (z-a)X(z) + bY(z) &= 8 \\ cX(z) + (z-d)Y(z) &= 3 \end{aligned}$$

for some $a, b, c, d \in \mathbb{R}$. Find their value.

(b) Solving the above linear system for $X(z)$ and $Y(z)$ we find

$$X(z) = \frac{e}{z+1} + \frac{f}{z-4}, \quad Y(z) = \frac{g}{z+1} + \frac{h}{z-4}.$$

for some $e, f, g, h \in \mathbb{R}$. Find their value.

(c) Conclude by finding an expression for $x(t)$ and $y(t)$.

Answer.

(a) Apply Laplace transform to get

$$zX(z) - \underbrace{x(0)}_{=8} = 2X(z) - 3Y(z) \implies (z-2)X(z) + 3Y(z) = 8,$$
$$zY(z) - \underbrace{y(0)}_{=3} = Y(z) - 2X(z) \implies 2X(z) + (z-1)Y(z) = 3,$$

i.e.

$$a = 2, \ b = 3, \ c = 2, \ d = 1.$$

(b) Solving the 2×2 system of equations for $X(z)$ and $Y(z)$, for example, using Cramer's rule we find

$$X(z) = \frac{\begin{vmatrix} 8 & 3 \\ 3 & z-1 \end{vmatrix}}{\begin{vmatrix} z-2 & 3 \\ 2 & z-1 \end{vmatrix}} = \frac{8z-17}{z^2-3z-4} = \frac{8z-17}{(z+1)(z-4)} = \frac{5}{z+1} + \frac{3}{z-4}$$

$$Y(z) = \frac{\begin{vmatrix} z-2 & 8 \\ 2 & 3 \end{vmatrix}}{\begin{vmatrix} z-2 & 3 \\ 2 & z-1 \end{vmatrix}} = \frac{3z-22}{z^2-3z-4} = \frac{3z-22}{(z+1)(z-4)} = \frac{5}{z+1} - \frac{2}{z-4},$$

i.e.

$$e = 5, \ f = 3, \ g = 5, \ h = -2. \tag{20}$$

(c) Using the previous result and applying the Laplace transform yields

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}(X) = 5e^{-t} + 3e^{4t} \\ y(t) &= \mathcal{L}^{-1}(Y) = 5e^{-t} - 2e^{4t}, \end{aligned}$$

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