MATH-207(d) Analysis IV
Exercise session 10

Exercice 1. Compute the following integral

$$\int_0^{2\pi} \frac{1}{\sqrt{5}-\sin\theta}\,\mathrm{d}\theta.$$

Answer. Using the identity $\sin\theta = -i(e^{i\theta}-e^{-i\theta})/2$ and the parameterisation of the unit circle $[0,2\pi) \ni \theta \mapsto \gamma(\theta) = e^{i\theta}$, we can write

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{\sqrt{5}-\sin\theta}\,\mathrm{d}\theta &= \int_0^{2\pi} \frac{1}{\sqrt{5}-\frac{1}{2i}(e^{i\theta}-e^{-i\theta})}\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{2i}{2\sqrt{5}i-(e^{i\theta}-e^{-i\theta})}\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{2i}{2\sqrt{5}i-e^{i\theta}+e^{-i\theta}}\frac{ie^{i\theta}}{ie^{i\theta}}\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{2}{2\sqrt{5}ie^{i\theta}-e^{2i\theta}+1} ie^{i\theta}\,\mathrm{d}\theta \\
&= \int_\gamma \frac{2}{2\sqrt{5}zi-z^2+1}\,\mathrm{d}z \\
&= \int_\gamma \frac{-2}{z^2-2\sqrt{5}zi-1}\,\mathrm{d}z \\
&= \int_\gamma \frac{-2}{(z-i(\sqrt{5}+2))(z-i(\sqrt{5}-2))}\,\mathrm{d}z.
\end{aligned}$$

Let the function f be defined as

$$f(z) = \frac{-2}{(z-i(\sqrt{5}+2))(z-i(\sqrt{5}-2))} \quad \text{for all } z \neq i(\sqrt{5}\pm 2).$$

Clearly, the only singularity of f inside the unit disc is located at $z_0 = i(\sqrt{5}-2)$. Moreover, this singularity is a simple pole with residue given by

$$\begin{aligned}
\mathrm{Res}_{i(\sqrt{5}-2)}(f) &= \lim_{z\to i(\sqrt{5}-2)} \frac{-2}{z-i(\sqrt{5}+2)} \\
&= \frac{-2}{i(\sqrt{5}-2)-i(\sqrt{5}+2)} = \frac{-2}{-4i} = \frac{1}{2i}.
\end{aligned}$$

We therefore conclude by the residue theorem that

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{\sqrt{5}-\sin\theta}\,\mathrm{d}\theta &= \int_\gamma \frac{2}{(z+i(\sqrt{5}+2))(z+i(\sqrt{5}-2))}\,\mathrm{d}z \\
&= \int_\gamma f(z)\,\mathrm{d}z = 2\pi i\cdot \mathrm{Res}_{i(\sqrt{5}-2)}(f) = \pi.
\end{aligned}$$

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Exercice 2. (a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$\int_{\mathbb{R}} |f(x)| \, \mathrm{d}x < \infty \qquad \text{and} \qquad \int_{\mathbb{R}} |xf(x)| \, \mathrm{d}x < \infty,$$

and define the function $g(x) = xf(x)$ for all $x \in \mathbb{R}$. Show that

$$\widehat{g}(\alpha) = i\frac{d}{d\alpha}\widehat{f}(\alpha).$$

(b) Using the above result, compute the Fourier transform of the function

$$x \mapsto g(x) = xe^{-|x|}.$$

Hint: Try to apply the inverse Fourier transform and make use of the integral

$$\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{1+x^2} \, \mathrm{d}x,$$

which has already been computed in a previous exercise sheet.

Answer.

(a) Notice that

$$\frac{d}{d\alpha}e^{-i\alpha x} = -ixe^{-i\alpha x}. \tag{1}$$

By definition, the Fourier transform of g is given by

$$\begin{aligned}
\hat{g}(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} xf(x)e^{-i\alpha x} \, \mathrm{d}x \\
&= \frac{1}{-i\sqrt{2\pi}} \int_{-\infty}^{\infty} -ixf(x)e^{-i\alpha x} \, \mathrm{d}x \\
&= \frac{1}{-i\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\frac{d}{d\alpha}e^{-i\alpha x} \, \mathrm{d}x = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\frac{d}{d\alpha}e^{-i\alpha x} \, \mathrm{d}x.
\end{aligned}$$

We want to pull out the derivative from the integral. We observe that

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\frac{d}{d\alpha}e^{-i\alpha x} \, \mathrm{d}x &= \lim_{L\to\infty} \int_{-L}^{L} f(x)\frac{d}{d\alpha}e^{-i\alpha x} \, \mathrm{d}x \\
&= \lim_{L\to\infty} \frac{d}{d\alpha} \int_{-L}^{L} f(x)e^{-i\alpha x} \, \mathrm{d}x \\
&= \frac{d}{d\alpha} \lim_{L\to\infty} \int_{-L}^{L} f(x)e^{-i\alpha x} \, \mathrm{d}x \\
&= \frac{d}{d\alpha} \int_{-\infty}^{\infty} f(x)e^{-i\alpha x} \, \mathrm{d}x
\end{aligned}$$

In combination,

$$\hat{g}(\alpha) = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\frac{d}{d\alpha}e^{-i\alpha x} \, \mathrm{d}x = \frac{i}{\sqrt{2\pi}}\frac{d}{d\alpha} \int_{-\infty}^{\infty} f(x)e^{-i\alpha x} \, \mathrm{d}x = i\frac{d}{d\alpha}\widehat{f}(\alpha).$$

We should emphasise here that in order to be completely rigorous, the interchange of the limit and the derivative with respect to α has to be mathematically justified. Exact conditions under which the limit and the derivative can be interchanged are beyond the scope of the current lecture. For the current problem however, the hypotheses that

$$\int_{\mathbb{R}} |f(x)| \, \mathrm{d}x < \infty \qquad \text{and} \qquad \int_{\mathbb{R}} |xf(x)| \, \mathrm{d}x < \infty$$

together with continuity of f consitute sufficient conditions to justify the interchange.

(b) Let us define the function $f\colon \mathbb{R} \to \mathbb{R}$ via

$$f(x) = e^{-|x|}$$

First, the Fourier transform of $g(x) = xf(x)$ satisfies

$$\widehat{g}(\alpha) = i\frac{d}{d\alpha}\widehat{f}(\alpha).$$

Next, we want to find the Fourier transform of f. From the results of the previous exercise sheet, we also know that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{1+\alpha^2} \, \mathrm{d}\alpha = \frac{1}{\sqrt{2\pi}}\pi e^{-|x|} := h(x).$$

As this identity shows, the last expression is the inverse Fourier transform of the function $\widehat{h}\colon \mathbb{R} \to \mathbb{R}$ given by

$$\widehat{h}(\alpha) = \frac{1}{1+\alpha^2}.$$

Consequently,

$$f(x) = e^{-|x|} = \frac{\sqrt{2\pi}}{\pi}h(x).$$

It follows that

$$\widehat{f}(\alpha) = \frac{\sqrt{2\pi}}{\pi}\widehat{h}(\alpha) = \frac{\sqrt{2\pi}}{\pi}\frac{1}{1+\alpha^2}.$$

We conclude that

$$\widehat{g}(\alpha) = i\frac{d}{d\alpha}\widehat{f}(\alpha) = -2i\frac{\sqrt{2\pi}}{\pi}\frac{\alpha}{(1+\alpha^2)^2} = -i\frac{2\sqrt{2}}{\sqrt{\pi}}\frac{\alpha}{(1+\alpha^2)^2}.$$

This is the desired result.

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Exercice 3. Let $\sigma \neq 0$ be a real number and let $f\colon \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = e^{-\frac{1}{2\sigma^2}x^2}.$$

The goal of this task is to compute the Fourier transform of the Gaussian function f.

(a) Show that

$$\widehat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}x^2} e^{-i\alpha x} \, \mathrm{d}x = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\alpha^2\sigma^2} \lim_{L\to\infty} \int_{-L+i\alpha\sigma^2}^{L+i\alpha\sigma^2} e^{-\frac{1}{2\sigma^2}z^2} \, \mathrm{d}z.$$

(b) Use results from complex analysis to establish that

$$\lim_{L\to\infty} \int_{-L+i\alpha\sigma^2}^{L+i\alpha\sigma^2} e^{-\frac{1}{2\sigma^2}z^2} \, \mathrm{d}z = \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}x^2} \, \mathrm{d}x$$

(c) Use polar coordinates to compute the integral

$$\left(\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}x^2} \, \mathrm{d}x \right)^2.$$

(d) Use the results of the above computation to finally deduce the Fourier transform $\widehat{f}$.

Answer.

(a) By the definition of the Fourier transform, we have

$$\begin{aligned}
\widehat{f}(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}x^2} e^{-i\alpha x} \, \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}x^2 - i\alpha x} \, \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}x^2 - i\alpha x + \frac{1}{2}\sigma^2\alpha^2 - \frac{1}{2}\sigma^2\alpha^2} \, \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x+i\alpha\sigma^2)^2 - \frac{1}{2}\sigma^2\alpha^2} \, \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\sigma^2\alpha^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x+i\alpha\sigma^2)^2} \, \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\sigma^2\alpha^2} \lim_{L\to\infty} \int_{-L}^{L} e^{-\frac{1}{2\sigma^2}(x+i\alpha\sigma^2)^2} \, \mathrm{d}x.
\end{aligned}$$

We now use the change of variables $z = x + i\alpha\sigma^2$ to obtain

$$\begin{aligned}
\widehat{f}(\alpha) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\sigma^2\alpha^2} \lim_{L\to\infty} \int_{-L}^{L} e^{-\frac{1}{2\sigma^2}(x-i\alpha\sigma^2)^2} \, \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\sigma^2\alpha^2} \lim_{L\to\infty} \int_{-L+i\alpha\sigma^2}^{L+i\alpha\sigma^2} e^{-\frac{1}{2\sigma^2}z^2} \, \mathrm{d}z.
\end{aligned}$$

(b) The first limit of integrals expresses an integral along the shifted real line $i\alpha\sigma^2 + \mathbb{R}$. We want to show that this agrees with the integral over $\mathbb{R}$. Consider the rectangle $\mathcal{R}^{(L)}$ in the complex plane defined as

$$\mathcal{R}^{(L)} = \mathcal{R}_1^{(L)} \cup \mathcal{R}_2^{(L)} \cup \mathcal{R}_3^{(L)} \cup \mathcal{R}_4^{(L)} \quad \text{where}$$

$$\begin{aligned}
\mathcal{R}_1^{(L)} &= \{z = x + i\alpha\sigma^2 \colon x \in [-L, L]\} \\
\mathcal{R}_2^{(L)} &= \{z = L + iy \colon y \in [0, \alpha\sigma^2]\} \\
\mathcal{R}_3^{(L)} &= \{z = x \colon a \in (-L, L)\} \\
\mathcal{R}_4^{(L)} &= \{z = -L + iy \colon y \in [0, \alpha\sigma^2]\}.
\end{aligned}$$

Obviously, $\mathcal{R}^{(L)}$ forms a closed simple differentiable curve for any choice of $L > 0$. Moreover the function $f\colon \mathbb{C} \to \mathbb{C}$ defined as

$$f(z) = e^{-\frac{1}{2\sigma^2}z^2}$$

is holomoprhic everywhere on $\mathbb{C}$. Consequently, by Cauchy's theorem, we deduce that

$$0 = \int_{\mathcal{R}^{(L)}} f(z)\,\mathrm{d}z = \int_{\mathcal{R}_1^{(L)}\cup\mathcal{R}_2^{(L)}\cup\mathcal{R}_3^{(L)}\cup\mathcal{R}_4^{(L)}} f(z)\,\mathrm{d}z.$$

Using now the definition of the rectangle $\mathcal{R}^{(L)}$ and the function f, we see that we must have

$$0 = \int_{-L+i\alpha\sigma^2}^{L+i\alpha\sigma^2} e^{-\frac{1}{2\sigma^2}z^2}\,\mathrm{d}z + \int_{L+\alpha\sigma^2}^{L} e^{-\frac{1}{2\sigma^2}z^2}\,\mathrm{d}z + \int_{L}^{-L} e^{-\frac{1}{2\sigma^2}z^2}\,\mathrm{d}z + \int_{-L}^{-L+\alpha\sigma^2} e^{-\frac{1}{2\sigma^2}z^2}\,\mathrm{d}z.$$

In particular,

$$\int_{-L+i\alpha\sigma^2}^{L+i\alpha\sigma^2} e^{-\frac{1}{2\sigma^2}z^2}\,\mathrm{d}z = \int_{-L}^{L} e^{-\frac{1}{2\sigma^2}z^2}\,\mathrm{d}z - \underbrace{\int_{L+i\alpha\sigma^2}^{L} e^{-\frac{1}{2\sigma^2}z^2}\,\mathrm{d}z}_{:=I_1} - \underbrace{\int_{-L}^{-L+i\alpha\sigma^2} e^{-\frac{1}{2\sigma^2}z^2}\,\mathrm{d}z}_{:=I_2}.$$

To obtain the desired result, we must argue that the integrals I_1 and I_2 approach zero in the limit $L \to \infty$. Notice that this integrals can be expressed as

$$I_1 = \int_{L+i\alpha\sigma^2}^{L} e^{-\frac{1}{2\sigma^2}z^2}\,\mathrm{d}z = \int_{t=1}^{t=0} e^{-\frac{1}{2\sigma^2}(L+it\alpha\sigma^2)^2} i\alpha\sigma^2\,\mathrm{d}t \quad \text{and}$$
$$I_2 = \int_{-L}^{-L+i\alpha\sigma^2} e^{-\frac{1}{2\sigma^2}z^2}\,\mathrm{d}z = \int_{t=0}^{t=1} e^{-\frac{1}{2\sigma^2}(-L+it\alpha\sigma^2)^2} i\alpha\sigma^2\,\mathrm{d}t$$

Before we proceed, take a look at the exponents in those integrals: as L gets very, very large, the exponents will be $-L^2/(2\sigma^2)$ and some smaller pertubation of that. As L grows to infinity, the exponentials will decay towards zero.

To make this observation more formal, notice that if $z \in \mathbb{C}$ is a complex number of the form $z = \pm L + it\alpha\sigma^2$ for $t \in [0, 1]$ then it holds that

$$\begin{aligned}\left|e^{-\frac{1}{2\sigma^2}z^2}\right| = \left|e^{-\frac{1}{2\sigma^2}(L^2 - t^2\alpha^2\sigma^4 \pm 2it\alpha L\sigma^2)}\right| &= e^{-\frac{1}{2\sigma^2}(L^2 - t^2\alpha^2\sigma^4)} \\ &= e^{-\frac{1}{2\sigma^2}L^2} e^{\frac{1}{2\sigma^2}t^2\alpha^2\sigma^4} \\ &\le e^{-\frac{1}{2\sigma^2}L^2} e^{\frac{1}{2\sigma^2}\alpha^2\sigma^4}.\end{aligned}$$

Consequently, we can deduce that

$$|I_1| \le e^{-\frac{1}{2\sigma^2}L^2} e^{\frac{1}{2\sigma^2}\alpha^2\sigma^4} \int_0^1 i\alpha\sigma^2\,\mathrm{d}t = e^{-\frac{1}{2\sigma^2}L^2} e^{\frac{1}{2\sigma^2}\alpha^2\sigma^4} i\alpha\sigma^2$$
$$|I_2| \le e^{-\frac{1}{2\sigma^2}L^2} e^{\frac{1}{2\sigma^2}\alpha^2\sigma^4} \int_0^1 i\alpha\sigma^2\,\mathrm{d}t = e^{-\frac{1}{2\sigma^2}L^2} e^{\frac{1}{2\sigma^2}\alpha^2\sigma^4} i\alpha\sigma^2.$$

In particular, $\lim_{L\to\infty} I_1 = \lim_{L\to\infty} I_2 = 0$, as claimed. We conclude that

$$\lim_{L\to\infty} \int_{-L+i\alpha\sigma^2}^{L+i\alpha\sigma^2} e^{-\frac{1}{2\sigma^2}z^2}\,\mathrm{d}z = \lim_{L\to\infty} \int_{-L}^{L} e^{-\frac{1}{2\sigma^2}z^2}\,\mathrm{d}z = \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}x^2}\,\mathrm{d}x.$$

(c) We make the simple observation

$$\left(\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}x^2}\,\mathrm{d}x\right)^2 = \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}x^2}\,\mathrm{d}x \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}y^2}\,\mathrm{d}y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x^2+y^2)}\,\mathrm{d}x\mathrm{d}y.$$

This is a radial integral. We can now use the polar coordiantes change of variables $x = r\cos(\theta)$ and $y = r\sin(\theta)$ to deduce that the above two-dimensional integral simplifies to

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x^2+y^2)}\,\mathrm{d}x\mathrm{d}y = \int_{r=0}^{r=\infty}\int_{\theta=0}^{\theta=2\pi} re^{-\frac{1}{2\sigma^2}(r^2)}\,\mathrm{d}\theta\mathrm{d}r &= 2\pi\int_{r=0}^{r=\infty} re^{-\frac{1}{2\sigma^2}(r^2)}\,\mathrm{d}r \\ &= -2\pi\left[\sigma^2 e^{-\frac{1}{2\sigma^2}r^2}\right]_{r=0}^{r=\infty} \\ &= 2\pi\sigma^2.\end{aligned}$$

Here, we have used the fact that

$$-\sigma^2\frac{d}{dr}e^{-\frac{1}{2\sigma^2}r^2} = re^{-\frac{1}{2\sigma^2}r^2}.$$

We thus conclude that

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}x^2}\,\mathrm{d}x = \sqrt{2\pi}|\sigma|.$$

(d) Collecting all of the above results, we finally obtain that the Fourier transform $\widehat{f}$ is given by

$$\begin{aligned}\widehat{f}(\alpha) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}x^2}e^{-i\alpha x}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\sigma^2\alpha^2}\lim_{L\to\infty}\int_{-L+i\alpha\sigma^2}^{L+i\alpha\sigma^2} e^{-\frac{1}{2\sigma^2}z^2}\,\mathrm{d}z \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\sigma^2\alpha^2}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}x^2}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\sigma^2\alpha^2}\sqrt{2\pi}|\sigma| = |\sigma|e^{-\frac{1}{2}\sigma^2\alpha^2}.\end{aligned}$$

Remark: We have discovered the very cool fact that the Fourier transform of a Gaussian is also a Gaussian! Not only this, we have discovered that the parameter σ that appears within the exponential and measures the variance of the Gaussian is *inverted* in the Fourier transform. This means that if the original Gaussian has a very small variance, then the Fourier transform Gaussian has a huge variance! This fact is closely related to the famous Heisenberg uncertainty principle in quantum physics.

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Exercice 4. Use the various properties of the Fourier transform given in Theorem 15.2 of the textbook together with the table of Fourier transforms given below to compute the following Fourier and inverse Fourier transforms.

(a) Let $f\colon \mathbb{R}\to\mathbb{R}$ be defined as

$$f(x) = \frac{e^{ix}}{\beta^2+\sigma^2x^2}, \quad \beta,\sigma \neq 0.$$

Compute the Fourier transform $\widehat{f}$.

(b) Let $g\colon \mathbb{R}\to\mathbb{R}$ be defined as

$$g(x) = x^2 e^{-\beta^2x^2}, \quad \beta \neq 0.$$

Compute the Fourier transform $\widehat{g}$.

(c) Let $h\colon \mathbb{R}\to\mathbb{R}$ be defined as

$$h(x) = \frac{x^2-2x+1}{(x^2-2x+2)^2}.$$

Compute the Fourier transform $\widehat{h}$. *Hint:* Find a relationship between the Fourier transform of an integrable function $f(\cdot)$ and the Fourier transform of the translated function $f(\cdot - c_0)$ for any $c_0 \in \mathbb{R}$.

Answer.

(a) Notice that we can write f in the form

$$f(x) = e^{ix}g(\sigma x), \quad \text{where}$$
$$g(x) = \frac{1}{\beta^2+x^2}.$$

Consequently, by taking advantage of the dilation and phase properties of the Fourier transform (see Property (v) Theorem 15.2 of the course textbook), we deduce that

$$\widehat{f}(\alpha) = \frac{1}{|\sigma|}\widehat{g}\left(\frac{\alpha-1}{\sigma}\right).$$

An expression for the Fourier transform of the function g can be found in the table (see Entry 6). We conclude that

$$\widehat{f}(\alpha) = \frac{1}{|\sigma\beta|}\sqrt{\frac{\pi}{2}}e^{-|\beta(\alpha-1)/\sigma|}.$$

(b) The trick is to use the differentiation property of the Fourier Transform as demonstrated in Exercise 2 above (alternatively, see Property (iv) Theorem 15.2 of the course textbook). We define

$$f_0(x) = e^{-\beta^2x^2},$$

Now,

$$\begin{aligned}\frac{d}{dx}f_0(x) &= -2\beta^2xe^{-\beta^2x^2},\\ \frac{d^2}{d^2x}f_0(x) &= -2\beta^2e^{-\beta^2x^2} + 4\beta^4x^2e^{-\beta^2x^2}\\ &= -2\beta^2f_0(x) + 4\beta^4g(x).\end{aligned}$$

We take the Fourier transform on both sides:

$$(i\alpha)^2 \hat{f}_0(x) = -2\beta^2 \hat{f}_0(x) + 4\beta^4 \hat{g}(x)$$

Isolating $\hat{g}$, we find

$$\left(2\beta^2 - \alpha^2\right) \hat{f}_0(x) = 4\beta^4 \hat{g}(x)$$

$$\hat{g}(x) = \frac{2\beta^2 - \alpha^2}{4\beta^4} \hat{f}_0(x).$$

It remains to find the Fourier transform of f_0, which can be read from the table (see Entry 9) or obtained using the results of Exercise 3 above. This brings:

$$\hat{g}(x) = \frac{2\beta^2 - \alpha^2}{4\beta^4} \hat{f}_0(x) = \frac{2\beta^2 - \alpha^2}{4\beta^4} \frac{e^{-\frac{\alpha^2}{4\beta^2}}}{\sqrt{2}|\beta|}.$$

(c) Following the hint provided in the question, let $f\colon \mathbb{R} \to \mathbb{R}$ be an integrable function, let $c_0 \in \mathbb{R}$ be any real number, and define $g\colon \mathbb{R} \to \mathbb{R}$ as

$$g(x) = f(x - c_0).$$

Then the Fourier transform of g is given by

$$\widehat{g}(\alpha) = \int_{-\infty}^{\infty} g(x) e^{-i\alpha x}\, \mathrm{d}x = \int_{-\infty}^{\infty} f(x - c_0) e^{-i\alpha x}\, \mathrm{d}x.$$

Using the change of variables $y = x - c_0$, we obtain

$$\widehat{g}(\alpha) = \int_{-\infty}^{\infty} f(y) e^{-i\alpha(y+c_0)}\, \mathrm{d}x = e^{-i\alpha c_0} \int_{-\infty}^{\infty} f(y) e^{-i\alpha y}\, \mathrm{d}x = e^{-i\alpha c_0} \widehat{f}(\alpha). \tag{2}$$

Returning now to the Fourier transform of the function h, we see that we can write

$$h(x) = \frac{(x-1)^2}{(1+(x-1)^2)^2} = g(x-1) \quad \text{with} \quad g(x) = \frac{x^2}{(1+x^2)^2}.$$

The Fourier transform of the function g can now be read from the table (see Entry 10). We therefore deduce that

$$\widehat{h}(\alpha) = e^{-i\alpha} \widehat{g}(\alpha) = \frac{1}{4} e^{-i\alpha} \sqrt{2\pi}\, (1 - |\alpha|)\, e^{-|\alpha|}.$$

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