MATH-207(d) Analysis IV
Exercise session 6

Exercice 1. For each of the following functions compute the Laurent series in the given z_0, determine its region of convergence, specify the nature of the singularity and report the residue.

(a) $f(z) = z\cos\left(\frac{1}{z}\right)$ in $z_0 = 0$.

(b) $f(z) = e^{1/z}\sin\left(\frac{1}{z}\right)$ in $z_0 = 0$

(c) $f(z) = \frac{e^z}{(z-1)^2}$ in $z_0 = 1$

(d) $f(z) = \frac{\sin z}{(z-\pi)^2}$ in $z_0 = \pi$.

(e) $f(z) = \frac{\sqrt{z}}{(z-1)^2}$ in $z_0 = 1$

Answer.

(a) The Taylor expansion of the cosine around 0 is the series

$$\cos y = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} y^{2n} = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \dots,$$

valid for all $y \in \mathbb{C}$. Using the replacement $y = \frac{1}{z}$, where $z \in \mathbb{C}$ is not zero, we get

$$z\cos\left(\frac{1}{z}\right) = z\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{z}\right)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\frac{1}{z^{2n-1}}$$
$$= z - \frac{1}{2}z^{-1} + \frac{1}{24}z^{-3} - \dots,,$$

valid for all $z \in \mathbb{C} \setminus \{0\}$. Hence, the Laurent series for f converges for all $z \in \mathbb{C}\setminus\{0\}$, i.e. the radius of convergence is infinity. Very explicitly, the coefficients are

$$c_n = \begin{cases} 1 & n = 1 \\ 1/(-n+1)! & n \text{ negative odd} \\ 0 & \text{otherwise.} \end{cases}$$

Regarding the nature of singularity $z = 0$, since there are infinitely many terms with negative powers of z in the Laurent series we conclude that this singularity is essential.

The residue is equal to the the coefficient in front of the z^{-1} term of the Laurent series: $c_{-1} = -\frac{1}{2}$ (obtained for $n = 1$).

(b) We can use the Taylor expansions around $y = 0$ for the functions

$$e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}, \quad \sin(y) = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{(2n+1)!}.$$

With $y = 1/z$ to obtain:

$$f(z) = e^{1/z} \sin\left(\frac{1}{z}\right) = \left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \dots\right)\left(\frac{1}{z} - \frac{1}{3!z^3} + \dots\right) = \frac{1}{z} + \frac{1}{z^2} + \dots$$

Alternatively, we use the Taylor expansion of the product $g(y) = e^y \sin(y)$ around $y = 0$. One observes

$$\begin{aligned} g(y) = e^y \sin(y) &= e^y \frac{e^{iy} - e^{-iy}}{2i} \\ &= \frac{e^{(1+i)y} - e^{(1-i)y}}{2i} = \frac{e^{(1+i)y}}{2i} - \frac{e^{(1-i)y}}{2i}. \end{aligned}$$

The product's derivatives are

$$g^{(n)}(y) = (1+i)^n \frac{e^{(1+i)y}}{2i} - (1-i)^n \frac{e^{(1-i)y}}{2i}.$$

The Taylor series $y = 0$ thus equals

$$g(y) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{(1+i)^n}{2i} - \frac{(1-i)^n}{2i}\right) y^n.$$

We substitute $y = \frac{1}{z}$ whenever z is non-zero. Thus:

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{(1+i)^n}{2i} - \frac{(1-i)^n}{2i}\right) z^{-n}.$$

Very explicitly, the coefficients are

$$c_n = \begin{cases} 0 & n > 0 \\ \dfrac{1}{(-n)!}\left(\dfrac{(1+i)^{-n}}{2i} - \dfrac{(1-i)^{-n}}{2i}\right) & n \leq 0 \end{cases}$$

Since there are infinitely many summands with negative powers, we conclude that $z_0 = 0$ is an essential singularity. The residue is equal to 1 (coefficient with $1/z$ term) and the series is convergent for all $z \in \mathbb{C}\backslash\{0\}$. (0 is the only singularity, and the Taylor series used are convergent for all y).

(c) One way of solving this problem uses the Taylor expansion of the exponential function around $z_0 = 1$:

$$e^z = \sum_{n=0}^{\infty} \frac{e}{n!} (z-1)^n.$$

Thus,

$$\begin{aligned} \frac{e^z}{(z-1)^2} = \frac{e}{(z-1)^2} \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} &= e \sum_{n=0}^{\infty} \frac{(z-1)^{n-2}}{n!} \\ &= \frac{e}{(z-1)^2} + \frac{e}{(z-1)} + \sum_{n=0}^{\infty} \frac{e}{(n+2)!}(z-1)^n. \end{aligned}$$

The Taylor series for e^y converges for all $y \in \mathbb{C}$. Therefore, for the Laurent series of f we must only exclude the singularity $z = 1$. Hence, the Laurent series for f converges for all $z \in \mathbb{C}\backslash\{1\}$, i.e., the radius of convergence is infinity.

Regarding the nature of singularity $z = 1$, since the highest negative power of $(z-1)$ is 2, we conclude that this is the pole of order 2.

Regarding the residue c_{-1}, it is equal to coefficient with term $(z-1)^{-1}$ of Laurent series which is e.

(d) We know the Taylor expansion

$$\sin y = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} y^{2n+1},$$

valid for all $y \in \mathbb{C}$. We substitute $y = z - \pi$, which yields

$$\sin(z-\pi) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (z-\pi)^{2n+1}.$$

We get

$$f(z) = \frac{\sin(z)}{(z-\pi)^2} = -\frac{\sin(z-\pi)}{(z-\pi)^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (z-\pi)^{2n-1}$$
$$= \frac{(z-\pi)^{-1}}{1!} + \frac{(z-\pi)^1}{3!} + \frac{(z-\pi)^3}{5!} + \dots.$$

From the expansion, we conclude that $z_0 = \pi$ is pole of order 1 and $c_{-1} = -1$. Since the Taylor series converges for all y, the region of the convergence for Laurent series of f is $\mathbb{C}\backslash\{\pi\}$.

(e) We will first find the Taylor expansion for $g(z) = \sqrt{z}$ at $z_0 = 1$. Since the coefficients of the Taylor expansion are $\frac{g^{(n)}(z_0)}{n!}$, we must determine the n-th derivative. We remember that $g(z) = \sqrt{z} = e^{(1/2)\log z}$ is holomorphic on

$$\mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Re} z \le 0, \operatorname{Im} z = 0\}.$$

Hence all the derivatives in $z_0 = 1$ exist. We get

$$\begin{aligned}
g'(z) &= \frac{1}{2} z^{-1/2}, \\
g''(z) &= -\frac{1}{4} z^{-3/2}, \\
g'''(z) &= \frac{3}{8} z^{-5/2}, \\
g^{(iv)}(z) &= -\frac{15}{16} z^{-7/2}, \\
&\vdots \\
g^{(n)}(z) &= (-1)^{n+1} \frac{1 \times 3 \times 5 \times \cdots \times (2n-3)}{2^n} z^{-(2n-1)/2}, \quad n = 2, 3, 4, \dots
\end{aligned}$$

Therefore, the expansion of $g(z)$ at $z_0 = 1$ is

$$g(z) = 1 + \frac{1}{2}(z-1) + \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1 \times 3 \times 5 \times \ldots \times (2n-3)}{2^n} \frac{(z-1)^n}{n!}$$

Finally, dividing by $(z-1)^2$ and rearranging the indices in the summation gives the Laurent series for f at $z_0 = 1$

$$\frac{\sqrt{z}}{(z-1)^2} = \frac{1}{(z-1)^2} + \frac{1}{2(z-1)} + \\ + \sum_{k=0}^{\infty} (-1)^{k+1} \frac{1 \times 3 \times 5 \times \ldots \times (2k+1)}{2^{k+2}(k+2)!} (z-1)^k.$$

To determine the convergence region for the Laurent series, first we must determine the region of convergence of Taylor series of g.

Recall that for the Taylor series to converge we need

$$|z-1| < \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{\frac{1\times3\times5\times\ldots\times(2n-3)}{2^n n!}}{\frac{1\times3\times5\times\ldots\times(2n-3)\times(2n-1)}{2^{n+1}(n+1)!}} \right|$$
$$= \lim_{n\to\infty} \frac{2n+2}{2n-1} = \lim_{n\to\infty} \left(1 + \frac{3}{2n-1}\right) = 1$$

Then, for f we also need to exclude the singularity $z = 1$. Thus, we have that the Laurent series converges for $0 < |z-1| < 1$, i.e. the radius of convergence is 1.

Alternatively, we can also argue that the Taylor series converges in the largest disk around z_0 over which $\sqrt{z}$ is holomorphic. That largest disk has radius 1. Again, this shows the radius of convergence 1.

Regarding the nature of singularity $z = 1$, since the highest negative power of $z - 1$ in the expansion is 2 we conclude that this is a pole of order 2.

Finally, inspecting the Laurent series we find $c_{-1} = \frac{1}{2}$.

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Exercice 2. Compute at least the singular part of the Laurent series of the following functions determine its region of convergence, specify the nature of the singularity and report the residue.

(a) $f(z) = \frac{\sin z}{\sin(z^2)}$ in $z_0 = 0$.

(b) $f(z) = \frac{1}{\cos^2\left(\frac{\pi}{2}z\right)}$ in $z_0 = 1$

(c) $f(z) = \frac{\log(1+z)}{\sin(z^2)}$ in $z_0 = 0$

(d) $f(z) = \frac{\sin z}{z(e^z-1)}$ in $z_0 = 0$

Answer. Ces exercices sont plus difficiles que les précédents.

(a) The numerator $\sin(z)$ and the denominator $\sin(z^2)$ are holomorphic close to $z_0 = 0$. Their Taylor series around 0 are already known:

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots$$
$$\sin(z^2) = z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \ldots$$

We observe

$$f(z) = \frac{1}{z} \underbrace{\frac{z\sin(z)}{\sin(z^2)}}_{=:g(z)}.$$

By construction,

$$g(z) := \frac{z^2 - \frac{z^4}{3!} + \frac{z^6}{5!} - \dots}{z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \dots} = \frac{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots}{1 - \frac{z^4}{3!} + \frac{z^8}{5!} - \dots}.$$

So the first non-zero coefficient of the Laurent series of g must be $g(0) = 1$. It follows that $f(z) = g(z)/z$ has a Laurent series at $z_0 = 0$ with first non-zero coefficient $c_{-1} = g(0) = 1$. In particular, it is a pole of order 1.

To determine the region of convergence, we have to exclude all other singularities of f. The additional singularities are

$$\sin\left(z^2\right) = 0 \quad \Longleftrightarrow \quad z = \pm\sqrt{k\pi}, \quad k \in \mathbb{Z}$$

The singularities closest to 0 are $\pm\sqrt{\pi}$ and $\pm\sqrt{-\pi} = \pm i\sqrt{\pi}$ which means that the Laurent series converges for all z satisfying $0 < |z| < \sqrt{\pi}$, i.e. the radius of convergence is $\sqrt{\pi}$.

(b) The denominator $\cos^2(\pi z/2)$ has a Taylor series around $z_0 = 1$. We do not know its coefficients yet and just write

$$g(z) := \cos^2(\pi z/2) = c_0 + c_1(z-1) + c_2(z-1)^2 + \dots$$

We will determine these terms as needed. We already know that $c_0 = g(1) = 0$ and

$$g'(z) = 2\cos(\pi z/2)(-1)\sin(\pi z/2)\frac{\pi}{2}, = -\pi \cdot \cos(\pi z/2)\sin(\pi z/2),$$
$$g''(z) = \frac{\pi^2}{2}\left(\sin^2\left(\frac{\pi z}{2}\right) - \cos^2\left(\frac{\pi z}{2}\right)\right).$$

Hence $c_1 = g'(1) = 0$ and $c_2 = g''(1) = \pi^2/2$. Hence

$$g(z) = c_2(z-1)^2 + c_3(z-1)^3 + \dots$$

It follows that

$$h(z) = \frac{(z-1)^2}{\cos^2(\pi z/2)} = \frac{(z-1)^2}{c_2(z-1)^2 + c_3(z-1)^3 + \dots} = \frac{1}{c_2 + c_3(z-1) + \dots}$$

is holomorphic at a neighborhood of $z_0 = 1$. But then $f(z) = h(z)/(z-1)^2$ must have a pole of order 2 at $z_0 = 1$.

To determine the radius of convergence, we need to find the largest disk around the singularity at $z_0 = 1$ that does not touch any other singularity. The singularities are the zeros of the denominator:

$$\cos\left(\frac{\pi}{2}z\right) = 0 \Longleftrightarrow \frac{\pi}{2}z = \frac{\pi}{2} + k\pi \Longleftrightarrow z = 1 + 2k, k \in \mathbb{Z}$$

Since the next singularities from the left and the right of $z_0 = 1$ are -1 and 3, the Laurent series converges for all z that satisfy $0 < |z-1| < 2$ (draw it in the complex plane, it will be more clear).

We determine the residual c_{-1} of f at z_0. That is the c_1 coefficient of h at z_0. We compute the derivative:

$$h'(z) = \frac{(z-1)^2}{\cos^2(\pi z/2)} = \frac{2(z-1)\cos^2\left(\frac{\pi}{2}z\right) + \pi(z-1)^2\cos\left(\frac{\pi}{2}z\right)\sin\left(\frac{\pi}{2}z\right)}{\cos^4\left(\frac{\pi}{2}z\right)}$$
$$= \frac{(z-1)}{\cos\left(\frac{\pi}{2}z\right)} \cdot \frac{2\cos\left(\frac{\pi}{2}z\right) + \pi(z-1)\sin\left(\frac{\pi}{2}z\right)}{\cos^2\left(\frac{\pi}{2}z\right)} = 0.$$

(c) This is similar to the Laurent series of this exercise. The numerator $\log(z+1)$ and the denominator $\sin(z^2)$ are holomorphic close to $z_0 = 0$. Their Taylor series around 0 are already known:

$$\log(z+1) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots$$
$$\sin(z^2) = z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \dots$$

We observe

$$f(z) = \frac{1}{z} \underbrace{\frac{z\log(z+1)}{\sin(z^2)}}_{=:g(z)}.$$

By construction,

$$g(z) := \frac{z^2 - \frac{z^3}{2} + \frac{z^4}{3} - \dots}{z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \dots} = \frac{1 - \frac{z}{2} + \frac{z^2}{3} - \dots}{1 - \frac{z^4}{3!} + \frac{z^8}{5!} - \dots}.$$

So the first non-zero coefficient of the Laurent series of g must be $g(0) = 1$. It follows that $f(z) = g(z)/z$ has a Laurent series at $z_0 = 0$ with first non-zero coefficient $c_{-1} = g(0) = 1$. In particular, it is a pole of order 1.

To determine the region of convergence, we have to exclude all other singularities of f. We already now from real analysis that the Taylor series of $\log(1+z)$ at $z_0 = 1$ has convergence radius 1. The singularities of the denominator are

$$\sin\left(z^2\right) = 0 \quad \Longleftrightarrow \quad z = \pm\sqrt{k\pi}, \quad k \in \mathbb{Z}$$

All these singularities are already further away from 1 than the singularity of the numerator. We conclude that the radius of convergence is 1.

(d) This is similar to the Laurent series of this exercise. The numerator $\sin(z+1)$ and the denominator $z(e^z - 1)$ are holomorphic close to $z_0 = 0$. Their Taylor series around 0 are already known:

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$
$$z(e^z - 1) = z\left(z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots\right) \qquad = z^2 + \frac{z^3}{2!} + \frac{z^4}{3!} + \dots$$

We observe

$$f(z) = \frac{1}{z} \underbrace{\frac{z\sin(z)}{z(e^z - 1)}}_{=:g(z)}.$$

By construction,

$$g(z) := \frac{z^2 - \frac{z^4}{3!} + \frac{z^6}{5!} - \dots}{z^2 + \frac{z^3}{2!} + \frac{z^4}{3!} + \dots} = \frac{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots}{1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots}.$$

So the first non-zero coefficient of the Laurent series of g must be $g(0) = 1$. It follows that $f(z) = g(z)/z$ has a Laurent series at $z_0 = 0$ with first non-zero coefficient $c_{-1} = g(0) = 1$. In particular, it is a pole of order 1.

To determine the region of convergence, we have to exclude all other singularities of f. The only zero of $1/z$ is $z_0 = 0$. The additional singularities of $e^z - 1$ are characterized by

$$e^z - 1 = 0 \iff e^z = 1 \iff z = \log(1) \iff z = 2\pi i k, \quad k \in \mathbb{Z}$$

The closest singularities to 0 are $\pm 2\pi i$ which means that the Laurent series converge for all z satisfying $0 < |z| < 2\pi$, i.e. the radius of the convergence is 2π.

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Exercice 3. Consider $f(z) = \frac{\sin(z^2+1)}{(z^2+1)^2}$.

(a) Find all singularities of f and determine their nature.

(b) Compute the residue in each singularity.

(c) Determine the region of convergence of the Laurent series around each singularity.

Answer.

(a) See below.

(b) The singularities are $z_0 = i$ and $z_0 = -i$. Everywhere else, the function is defined and complex differentiable. We develop the first few terms of Laurent series at each singularity.

First, consider $z_0 = i$. We integrate along a closed simple regular curve γ around z_0:

$$c_n = \frac{1}{2\pi i} \int_\gamma \frac{\sin(z^2 + 1)}{(z + i)^2 (z - i)^2 (z - i)^{n+1}}.$$

When $n \leq -3$, then the integrand is holomorphic at a (small) neighborhood of $z_0 = i$, and so the Cauchy integral theorem implies that $c_n = 0$. Hence this a pole of order at most 2. A similar argument shows that $z_0 = -i$ is a pole of order at most 2 as well.

Next, if $n = -2$, we integrate around $z_0 = i$:

$$\begin{aligned} c_{-2} &= \frac{1}{2\pi i} \int_\gamma \frac{\sin(z^2 + 1)}{(z + i)^2 (z - i)^2 (z - i)^{-1}} \\ &= \frac{1}{2\pi i} \int_\gamma \frac{\sin(z^2 + 1)/(z + i)^2}{(z - i)} \\ &= \frac{1}{2\pi i} \sin(i^2 + 1)/(i + i)^2 = 0. \end{aligned}$$

If $n = -1$, we also find that

$$\begin{aligned} c_{-1} &= \frac{1}{2\pi i}\int_\gamma \frac{\sin(z^2+1)/(z+i)^2}{(z-i)^2} \\ &= \left(\frac{\sin(z^2+1)}{(z+i)^2}\right)'(i) \\ &= \frac{-2\sin(z^2+1)+2z(z+i)\cos(z^2+1)}{(z+i)^3}(i) = \frac{2i(2i)\cos(0)}{(2i)^3} = \frac{1}{2i}. \end{aligned}$$

Consequently, this a pole of order 1, and the residual is

$$c_{-1} = \frac{1}{2i}.$$

Similarly, consider the case $z_0 = -i$. If $n = -2$, we once more find

$$\begin{aligned} c_{-2} &= \frac{1}{2\pi i}\int_\gamma \frac{\sin(z^2+1)}{(z+i)^2(z-i)^2(z+i)^{-1}} \\ &= \frac{1}{2\pi i}\int_\gamma \frac{\sin(z^2+1)/(z-i)^2}{(z+i)} \\ &= \frac{1}{2\pi i}\sin((-i)^2+1)/(-i-i)^2 = 0. \end{aligned}$$

If $n = -1$, we also find that

$$\begin{aligned} c_{-1} &= \frac{1}{2\pi i}\int_\gamma \frac{\sin(z^2+1)/(z+i)^2}{(z-i)^2} \\ &= \left(\frac{\sin(z^2+1)}{(z-i)^2}\right)'(-i) \\ &= \frac{-2\sin(z^2+1)+2z(z-i)\cos(z^2+1)}{(z-i)^3}(-i) = \frac{2(-i)(-2i)\cos(0)^2}{(-2i)^3} = \frac{-1}{2i}. \end{aligned}$$

Consequently, this a pole of order 1, and the residual is

$$c_{-1} = \frac{-1}{2i}.$$

(c) The distance between the singularites $-i$ and i is 2. We see that $0 < |z-i| < 2$ and $0 < |z+i| < 2$ are the regions of convergence of the Laurent series at i and $-i$, respectively.

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Exercice 4. Find the coefficients of the Laurent series of the following functions around the specified point. Determine the nature of the singularity.

(a) $g(z) = \dfrac{e^z}{(z-2)^2}$ and $z_0 = 2$

(b) $g(z) = \dfrac{2z^3+5z^2+z+\imath}{z+\imath}$ and $z_0 = -\imath$

(c) $g(z) = \dfrac{\cos\left((z-1)^2\right)}{(z-1)^3}$ at $z_0 = 1$

(d) $g(z) = \dfrac{1}{z(z-1)^2}$ and $z_0 = 1$

Answer.

(a) Since the complex exponential function is holomorphic on $\mathbb{C}$, we know that the Taylor series of $g(z) = e^z$ at $z_0 = 2$ coincides with the Laurent series at $z_0 = 2$. Moreover, this series is given by (see Exercise 1.)

$$g(z) = \sum_{n=0}^{\infty} \frac{e^2}{n!}(z-2)^n.$$

Consequently, the Laurent series of the function g at the point $z_0 = 2$ is given by

$$g(z) = \frac{1}{(z-2)^2} \sum_{n=0}^{\infty} \frac{e^2}{n!}(z-2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!}(z-2)^{n-2} = \sum_{n=-2}^{\infty} \frac{e^2}{(n+2)!}(z-2)^n.$$

The coefficients of the Laurent series are given by

$$a_n = \frac{e^2}{(n+2)!} \quad \forall n \geq -2 \qquad \text{and} \qquad a_n = 0 \;\; \text{otherwise}.$$

Consequently, f has a second order pole at $z_0 = 2$.

(b) Following the same strategy as before, we first introduce the holomorphic function $g(z) = 2z^3 + 5z^2 + z + \imath$. The Taylor series of this function at $z_0 = -\imath$ can easily be computed as

$$g(z) = -(5-2\imath) - (5+10\imath)(z+\imath) + (5-6\imath)(z+\imath)^2 + 2(z+\imath)^3.$$

Consequently, the Laurent series of the function g at the point $z_0 = -\imath$ is given by

$$\begin{aligned} g(z) &= \frac{-(5-2\imath) - (5+10\imath)(z+\imath) + (5-6\imath)(z+\imath)^2 + 2(z+\imath)^3}{z+\imath} \\ &= -\frac{(5-2\imath)}{z+\imath} - (5+10\imath) + (5-6\imath)(z+\imath) + 2(z+\imath)^2. \end{aligned}$$

The coefficients of the Laurent series are given by

$$a_{-1} = -(5-2\imath); \quad a_0 = -(5+10\imath); \quad a_1 = (5-6\imath); \quad a_2 = 2; \qquad a_n = 0 \;\; \text{otherwise}.$$

Consequently, g has a first order pole at $z_0 = -\imath$.

(c) As before, we introduce the holomorphic function $g(z) = \cos\big((z-1)^2\big)$. In order to compute the Taylor series of this function at $z_0 = 1$, we first compute the Taylor series of the auxiliary function $h(y) = \cos(y)$ at $y_0 = 0$, which is given by

$$h(y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} y^{2n}.$$

Using now the substitution $y = (z-1)^2$, we deduce that the Taylor series of the function $\cos\big((z-1)^2\big)$ at $z_0 = 1$ is given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(z-1)^{4n}.$$

Consequently, the Laurent series of the function g at the point $z_0 = 1$ is given by

$$\begin{aligned} g(z) &= \frac{1}{(z-1)^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (z-1)^{4n} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (z-1)^{4n-3}. \end{aligned}$$

We thus deduce that the Laurent coefficients are given by

$$a_{4n-3} = \frac{(-1)^n}{(2n)!} \quad \forall n \in \mathbb{N}, \qquad a_n = 0 \ \text{ otherwise.}$$

In particular, g has a third order pole at $z_0 = 1$.

(d) We introduce the function $g\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $g(z) = 1/z$. Of course g is not holomorphic on the entire complex plane but g is indeed holomorphic in the open ball of radius 1 centered at $z_0 = 1$. Let us denote this open ball by $\mathcal{B}_1(z_0)$.

It follows that in the ball $\mathcal{B}_1(z_0)$, the Taylor series of g at $z_0 = 1$ coincides with the Laurent series. To obtain this Taylor series, we note that

$$\frac{1}{z} = \frac{1}{(z-1)+1} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n,$$

and consequently

$$g(z) = \sum_{n=0}^{\infty} (-1)^n (z-1)^n \qquad \forall z \in \mathcal{B}_1(z_0).$$

Therefore, the Laurent series of the function g at the point $z_0 = 1$ is given by

$$\begin{aligned} g(z) &= \frac{1}{(z-1)^2} \sum_{n=0}^{\infty} (-1)^n (z-1)^n = \sum_{n=0}^{\infty} (-1)^n (z-1)^{n-2} \\ &= \sum_{n=-2}^{\infty} (-1)^n (z-1)^n \qquad \forall z \in \mathcal{B}_1(z_0) \setminus \{z_0\}. \end{aligned}$$

Studying the terms appearing in this Laurent series, we can deduce that the Laurent coefficients are given by

$$a_n = (-1)^n \ \ \forall n \in \{-2, -1\} \cup \mathbb{N}, \qquad a_n = 0 \ \text{ otherwise.}$$

In particular, g has a second order pole at $z_0 = 1$.

■

Exercice 5. Find the coefficients of the Laurent series of the following functions around the specified point. Determine the nature of the singularity.

(a) $h(z) = \sin\left(\frac{1}{z}\right)$ and $z_0 = 0$

(b) $h(z) = \sin\left((z-1)^{-1}\right)$ and $z_0 = 1$

(c) $h(z) = (z-2)^2 \cos\left((z-2)^{-4}\right)$ and $z_0 = 2$

(d) $h(z) = (z+\imath)^5 e^{\left((z+\imath)^{-2}\right)}$ and $z_0 = -\imath$

Answer.

(a) Notice that the function $f(y) = \sin(y)$ is holomorphic on the entire complex plane. The Taylor series of f at $y_0 = 0$ is given by

$$\sin(y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} y^{2n+1} \qquad \forall y \in \mathbb{C}.$$

Using now the substitution $y = 1/z$ for $z \neq 0$, we deduce that the Laurent series for $h(z) = \sin(1/z)$ at $z_0 = 0$ is given by

$$\sin(1/z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{-2n-1} = \sum_{n=-\infty}^{0} \frac{(-1)^n}{(-2n+1)!} z^{2n-1} \qquad \forall z \in \mathbb{C} \setminus \{0\}.$$

Consequently, the Laurent series coefficients are given by

$$a_{-2n-1} = \frac{(-1)^n}{(2n+1)!} \quad \forall n \in \mathbb{N}, \qquad a_n = 0 \;\text{ otherwise},$$

and h has an essential singularity at $z_0 = 0$.

(b) Recall once again that the function $f(y) = \sin(y)$ is holomorphic on the entire complex plane. The Taylor series of f at $y_0 = 0$ is given by

$$\sin(y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} y^{2n+1} \qquad \forall y \in \mathbb{C}.$$

This time, we use the substitution $y = (z-1)^{-1}$ for $z \neq 1$ to deduce that the Laurent series for $h(z) = \sin\left((z-1)^{-1}\right)$ at $z_0 = 1$ is given by

$$\sin\left((z-1)^{-1}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (z-1)^{-2n-1} = \sum_{n=-\infty}^{0} \frac{(-1)^n}{(-2n+1)!} z^{2n-1} \qquad \forall z \in \mathbb{C} \setminus \{1\}.$$

Consequently, the Laurent series coefficients are given by

$$a_{-n} = \frac{(-1)^n}{(2n+1)!} \quad \forall n \in \mathbb{N}, \qquad a_n = 0 \;\text{ otherwise},$$

and h has an essential singularity at $z_0 = 1$.

(c) As before, we observe that the function $f(y) = \cos(y)$ is holomorphic on the entire complex plane. The Taylor series of f at $y_0 = 0$ is given by

$$\cos(y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} y^{2n} \qquad \forall y \in \mathbb{C}.$$

Using now the substitution $y = (z-2)^{-4}$ for $z \neq 2$, we deduce that the Laurent series for $\cos\left((z-2)^{-4}\right)$ at $z_0 = 2$ is given by

$$\cos\left((z-2)^{-4}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (z-2)^{-8n} = \sum_{n=-\infty}^{0} \frac{(-1)^n}{(-2n)!} (z-2)^{8n} \qquad \forall z \in \mathbb{C} \setminus \{2\}.$$

As a consequence, the Laurent series for the function $h(z) = (z-2)^2 \cos\big((z-2)^{-4}\big)$ at $z_0 = 2$ is given by

$$\sum_{n=-\infty}^{0} \frac{(-1)^n}{(-2n)!}(z-2)^{8n+2} \qquad \forall z \in \mathbb{C} \setminus \{2\}.$$

Therefore, the Laurent series coefficients are given by

$$a_{-8n+2} = \frac{(-1)^n}{(2n)!} \quad \forall n \in \mathbb{N}, \qquad a_n = 0 \ \text{ otherwise},$$

and h has an essential singularity at $z_0 = 2$.

(d) Clearly, the function $f(y) = \exp(y)$ is holomorphic on the entire complex plane, and the Taylor series of f at $y_0 = 0$ is given by

$$\exp(y) = \sum_{n=0}^{\infty} \frac{1}{n!} y^n \qquad \forall y \in \mathbb{C}.$$

Using now the substitution $y = (z+\imath)^{-2}$ for $z \neq -\imath$, we deduce that the Laurent series for $\exp\big((z+\imath)^{-2}\big)$ at $z_0 = -\imath$ is given by

$$\exp\big((z+\imath)^{-2}\big) = \sum_{n=0}^{\infty} \frac{1}{n!}(z+\imath)^{-2n} = \sum_{n=-\infty}^{0} \frac{1}{(-n)!}(z+\imath)^{2n} \qquad \forall z \in \mathbb{C} \setminus \{-\imath\}.$$

As a consequence, the Laurent series for the function $h(z) = (z+\imath)^5 \exp\big((z+\imath)^{-2}\big)$ at $z_0 = -\imath$ is given by

$$\sum_{n=-\infty}^{0} \frac{1}{(-n)!}(z+\imath)^{2n+5} \qquad \forall z \in \mathbb{C} \setminus \{-\imath\}.$$

Therefore, the Laurent series coefficients are given by

$$a_{-2n+5} = \frac{(-1)^n}{n!} \quad \forall n \in \mathbb{N}, \qquad a_n = 0 \ \text{ otherwise},$$

and h has an essential singularity at $z_0 = 2$.

■

Exercice 6. (Extra) Prove Liouville's theorem: if $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded, that is, we have $|f(z)| \leq M$ for some $M \geq 0$, then f is constant. The following steps might be helpful.

(a) Write f as a power series with coefficients given by the Cauchy integral formula.

(b) Express the coefficients as line integrals over a circle of radius $r > 0$. Simplify the expression.

(c) Estimate the magnitude of the coefficients.

Answer. Suppose that f is holomorphic. We represent it by a Taylor series around $z_0 = 0$:

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

where $a_k \in \mathbb{C}$ are the coefficients. Whenever $\gamma : [0, 2\pi] \to \mathbb{C}$ is a closed curve that contains z_0, we can use Cauchy's integral formula:

$$a_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{2\pi i} \oint_\gamma \frac{f(\xi)}{\xi^{k+1}} \, d\xi.$$

We use a special choice of the curve, namely $\gamma(t) = re^{it}$ for some $r > 0$. Then we find

$$\begin{aligned}
|a_k| &= \frac{1}{2\pi} \left| \oint_\gamma \frac{f(\xi)}{\xi^{k+1}} \, d\xi \right| \\
&= \frac{1}{2\pi} \left| \oint_0^{2\pi} \frac{f(re^{it})}{r^{k+1} e^{(k+1)it}} \cdot ire^{it} \, dt \right| \\
&= \frac{1}{2\pi} \left| \oint_0^{2\pi} \frac{f(re^{it})}{r^{k+1} e^{(k+1)it}} \cdot re^{it} \, dt \right| = \frac{1}{2\pi} \left| \oint_0^{2\pi} \frac{f(re^{it})}{r^k e^{kit}} \, dt \right|.
\end{aligned}$$

Next, we use the following fact about integrals: given a (say, continuous) function $g : [a, b] \to \mathbb{C}$, one has the inequality

$$\left| \int_a^b g(t) dt \right| \le \int_a^b |g(t)| \, dt.$$

We can apply this and obtain:

$$\begin{aligned}
\frac{1}{2\pi} \left| \oint_0^{2\pi} \frac{f(re^{it})}{r^k e^{kit}} \, dt \right| &\le \frac{1}{2\pi} \oint_0^{2\pi} \left| \frac{f(re^{it})}{r^k e^{kit}} \right| \, dt \\
&\le \frac{1}{2\pi} \oint_0^{2\pi} \frac{|f(re^{it})|}{|r^k e^{kit}|} \, dt \\
&\le \frac{1}{2\pi} \oint_0^{2\pi} \frac{M}{|r^k e^{kit}|} \, dt \le \frac{M}{2\pi} \oint_0^{2\pi} \frac{1}{r^k} \, dt \le \frac{M}{r^k}.
\end{aligned}$$

As $r > 0$ was completely arbitrary, we can make it arbitrarily large. But that means that $|a_k|$ will be arbitrarily small, that is, equal zero.
This argument works for any $k \ge 1$. We conclude that the coefficients $a_1, a_2, a_3, \ldots$ must vanish. Hence, $f(z) = a_0$. ■