MATH-207(d) Analysis IV
Exercise session 4

1. Integrals on circles.
Let $\gamma_1 := \{z \in \mathbb{C} : |z| = 1\}$ be the circle in the complex plane with center 0 and radius 1 and $\gamma_2 := \{z \in \mathbb{C} : |z-2| = 1\}$ the circle with center 2 and radius 1. Compute the value of the following integrals, some of which have been discussed in the lecture.

(a) $\int_{\gamma_1} \frac{1}{z} \mathrm{d}z$.

(b) $\int_{\gamma_1} \frac{1}{z^2} \mathrm{d}z$.

(c) $\int_{\gamma_2} \frac{1}{z} \mathrm{d}z$.

(d) $\int_{\gamma_2} \frac{1}{z^2} \mathrm{d}z$.

Answer. Using the Cauchy-integral formula for $f(z) = 1$:

(a)
$$\int_{\gamma_1} \frac{1}{z} \mathrm{d}z = 2\pi i f(0) = 2\pi i$$

(b)
$$\int_{\gamma_1} \frac{1}{z^2} dz = \frac{2\pi i}{1!} f'(0) = 0$$

(c) Since $1/z$ is holomorphic on this circle and its interior we have
$$\int_{\gamma_2} \frac{1}{z} \mathrm{d}z = 0$$

(d) Since $1/z^2$ is holomorphic on this circle and its interior we have
$$\int_{\gamma_2} \frac{1}{z^2} \mathrm{d}z = 0$$

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2. More integrals on circles

Compute the following integrals.

(a) $\int_\gamma \frac{e^{2z}}{z}\mathrm{d}z$ with $\gamma = \{z \in \mathbb{C} : |z| = 2\}$.

(b) $\int_\gamma \frac{z^3+2z^2+2}{z-2i}\mathrm{d}z$ with $\gamma = \left\{z \in \mathbb{C} : |z-2i| = \frac{1}{4}\right\}$.

(c) $\int_\gamma \frac{\sin\left(2z^2+3z+1\right)}{z-\pi}\mathrm{d}z$ with $\gamma = \{z \in \mathbb{C} : |z-\pi| = 1\}$.

(d) $\int_\gamma \frac{3z^2+2z+\sin(z+1)}{(z-2)^2}\mathrm{d}z$ with $\gamma = \{z \in \mathbb{C} : |z-2| = 1\}$.

(e) $\int_\gamma \frac{e^z}{z(z+2)}\mathrm{d}z$ with $\gamma = \{z \in \mathbb{C} : |z| = 1\}$.

Answer.

(a) The function $f(\xi) = e^{2\xi}$ is holomorphic on $\mathbb{C}$. The curve γ is the circle centered at the origin of radius 2. Thus by Cauchy integral formula

$$\int_\gamma \frac{f(\xi)}{\xi}\mathrm{d}\xi = 2\pi i f(0) = 2\pi i.$$

Note that we could have tried to compute the integral directly, for example, parametrizing the circle as

$$\gamma(t) = 2e^{it}, \quad 0 \le t \le 2\pi.$$

In this case we would find

$$\int_\gamma \frac{e^{2z}}{z}\mathrm{d}z = \int_0^{2\pi} \frac{e^{4e^{it}}}{e^{it}} ie^{it}\mathrm{d}t = i\int_0^{2\pi} e^{4\cos t}e^{4i\sin t}\mathrm{d}t,$$

which is not trivial to compute.

(b) The function $f(\xi) = \xi^3 + 2\xi^2 + 2$, $z = 2i$ is holomorphic on $\mathbb{C}$ and the curve γ is the circle of center $2i$ and radius $\frac{1}{4}$. Thus by Cauchy integral formula

$$\int_\gamma \frac{z^3+2z^2+2}{z-2i}\mathrm{d}z = 2\pi i f(2i) = 16\pi - 12\pi i.$$

Although it cannot be done in one line, the computation without the Cauchy integral formula is tractable. Take for instance

$$\gamma(t) = 2i + \frac{1}{4}e^{it}, \quad 0 \le t \le 2\pi.$$

Then we would have to compute the following by parts.

$$\int_\gamma \frac{z^3+2z^2+2}{z-2i}\mathrm{d}z = \int_0^{2\pi} \frac{\left(2i+e^{it}/4\right)^3 + 2\left(2i+e^{it}/4\right)^2 + 2}{e^{it}/4}\frac{i}{4}e^{it}\mathrm{d}t.$$

(c) The function $f(\xi) = \sin\left(2\xi^2 + 3\xi + 1\right)$ is holomorphic on $\mathbb{C}$ and the curve γ is the circle centered in $z = \pi$ of radius 1. Then the Cauchy integral formula yields

$$\int_\gamma \frac{\sin\left(2z^2+3z+1\right)}{z-\pi}\mathrm{d}z = 2\pi i f(\pi) = 2\pi i \sin\left(2\pi^2 + 3\pi + 1\right).$$

(d) The function $f(\xi) = 3\xi^2 + 2\xi + \sin(\xi + 1)$ is holomorphic on $\mathbb{C}$ and the curve γ is the circle centered in $z = 2$ of radius 1. Thus to apply the Cauchy integral differentiation formula with $n = 1$ we compute $f'(\xi) = 6\xi + 2 + \cos(\xi + 1)$ and find

$$\int_\gamma \frac{3z^2 + 2z + \sin(z+1)}{(z-2)^2} \mathrm{d}z = 2\pi i f'(2) = 2\pi i (14 + \cos 3).$$

(e) The function $f(\xi) = e^\xi/(\xi + 2)$ is holomorphic on $\mathbb{C} \setminus \{-2\}$ and the curve γ is the unit circle centered at the origin, therefore f is holomorphic on $\overline{\mathrm{int}(\gamma)}$. By Cauchy integral formula we find

$$\int_\gamma \frac{e^z}{z(z+2)} \mathrm{d}z = 2\pi i f(0) = 2\pi i \frac{1}{2} = \pi i.$$

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3. Another integral on a closed curve.
Let γ be any simple, closed and piecewise regular curve. Discuss the value of

$$\int_\gamma \frac{5z^2 - 3z + 2}{(z-1)^3} \mathrm{d}z$$

depending on the curve γ. You must distinguish the cases:

- The pole of the integrand lies within the region encircled by the curve
- The pole of the integrand lies outside of the region encircled by the curve
- The pole lies on the curve

Answer. First note that the integrand is holomorphic on $\mathbb{C} \setminus \{1\}$.

(a) Case $1 \in \mathrm{int}(\gamma)$, that is, 1 lies in the region encircled by the curve γ. We apply the Cauchy integral formula to $f(\xi) = 5\xi^2 - 3\xi + 2$ in $z = 1$ and with $n = 2$. We have that $f''(1) = 10$ and thus

$$\int_\gamma \frac{5z^2 - 3z + 2}{(z-1)^3} \mathrm{d}z = 10\pi i.$$

(b) Case $1 \notin \overline{\mathrm{int}(\gamma)}$, that is, 1 lies outside that region, and is not on γ either[1]. The Cauchy theorem immediately allows to conclude

$$\int_\gamma \frac{5z^2 - 3z + 2}{(z-1)^3} \mathrm{d}z = 0.$$

(c) Case $1 \in \gamma$. The integral is ill-defined as the curve passes through the pole $z = 1$.

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[1] If you have discussed open sets in Analysis I & II, then may have seen the "closure" of a set $\overline{A}$.

4. Yet another integral.
Compute the integral

$$\int_\gamma \frac{e^{z^2}}{(z-1)^2\,(z^2+4)}\mathrm{d}z$$

in the following cases :

(a) γ is the circle centered in $z=1$ of radius 1.

(b) γ is the boundary of the rectangle $\left\{z \in \mathbb{C} : -\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2},\, 0 \leq \mathrm{Im}(z) \leq 4\right\}$.

(c) γ is the boundary of the rectangle $\{z \in \mathbb{C} : -2 \leq \mathrm{Re}(z) \leq 0,\, -1 \leq \mathrm{Im}(z) \leq 1\}$.

Answer.

(a) The function $f(\xi) = e^{\xi^2}/\,(\xi^2+4)$ is holomorphic on $\mathbb{C} \setminus \{2i, -2i\}$. The circle around 1 with radius 1 is within that set and does not include any of the two singularities. By Cauchy integral formula in $z=1$ for $n=1$ we get

$$\int_\gamma \frac{e^{z^2}}{(z-1)^2\,(z^2+4)}\mathrm{d}z = 2\pi i f'(1).$$

Differentiating f gives

$$f'(\xi) = \frac{2\xi e^{\xi^2}\,(\xi^2+4) - 2\xi e^{\xi^2}}{\left(\xi^2+4\right)^2}$$

and yields $f'(1) = 8e/25$. Thus we conclude

$$\int_\gamma \frac{e^{z^2}}{(z-1)^2\,(z^2+4)}\mathrm{d}z = \frac{16e\pi}{25}i$$

(b) Note that

$$\frac{e^{z^2}}{(z-1)^2\,(z^2+4)} = \frac{e^{z^2}}{(z-1)^2\,(z+2i)\,(z-2i)}$$

The function

$$f(\xi) = \frac{e^{\xi^2}}{(\xi-1)^2(\xi+2i)}$$

is holomorphic on $\mathbb{C} \setminus \{1, -2i\}$, and the rectangle does not include any of the two singularities. By Cauchy integral formula in $z=2i$ we get

$$\int_\gamma \frac{e^{z^2}}{(z-1)^2\,(z^2+4)}\mathrm{d}z = \int_\gamma \frac{f(z)}{(z-2i)}\mathrm{d}z = 2\pi i f(2i) = \frac{-\pi e^{-4}}{2(3+4i)}.$$

(c) In this last situation, the integrand is holomorphic on $\mathbb{C} \setminus \{1, 2i, -2i\}$. The curve lies within this set and does not contain any of the three singularities. Therefore, by the Cauchy theorem we immediately get

$$\int_\gamma \frac{e^{z^2}}{(z-1)^2\,(z^2+4)}\mathrm{d}z = 0.$$

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5. Difficult integrals made "easy".
Complex analysis can be a powerful tool to calculate complicated integrals, even if those integ7rals do not involve complex numbers at all! The goal of this exercise is to show that

$$\int_{-\infty}^{+\infty} e^{-x^2}\cos(2bx)\mathrm{d}x = \sqrt{\pi}e^{-b^2} \tag{1}$$

$$\int_{-\infty}^{+\infty} e^{-x^2}\sin(2bx)\mathrm{d}x = 0. \tag{2}$$

(a) Argue that $f(z) = e^{-z^2}$ is holomorphic on $\mathbb{C}$.

(b) Consider the path $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ shown in Figure 1

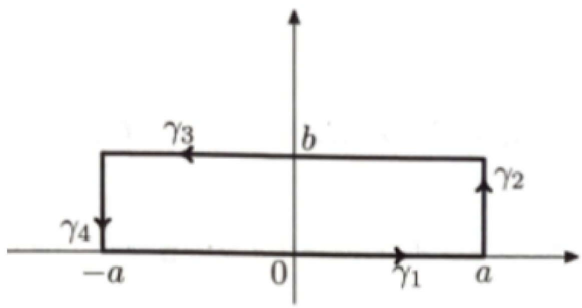

Figure 1: Path in complex plane for Exercise 4(b).

(i) Argue that $\int_\gamma f(z)\mathrm{d}z = 0$.

(ii) Show that $\lim\limits_{a\to+\infty} \int_{\gamma_2} f(z)\mathrm{d}z = \lim\limits_{a\to+\infty} \int_{\gamma_4} f(z)\mathrm{d}z = 0$.

(iii) Using that $\int_{-\infty}^{+\infty} e^{-x^2}\mathrm{d}x = \sqrt{\pi}$, conclude by showing (1) and (2).

Answer.

(a) The function f is the composition of holomorphic functions on $\mathbb{C}$ therefore f is also holomorphic on $\mathbb{C}$.

(b) (i) We have that γ is a simple, closed and piecewise regular path and f is holomorphic on $\overline{\mathrm{int}(\gamma)}$. By Cauchy theorem we find

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

(ii) Let us parametrize the segments as

$$\begin{aligned}
\gamma_1(t) &= t, && \text{with } t \in [-a, a]\,,\\
\gamma_2(t) &= a + it, && \text{with } t \in [0, b]\,,\\
\gamma_3(t) &= -t + ib, && \text{with } t \in [-a, a]\,,\\
\gamma_4(t) &= -a + i(b - t), && \text{with } t \in [0, b]\,,
\end{aligned}$$

Then with the contour integral formula we find

$$\int_{\gamma_2} f(z)\mathrm{d}z = i\int_0^b e^{-(a+it)^2}\mathrm{d}t = ie^{-a^2}\int_0^b e^{-2ait+t^2}\mathrm{d}t$$

and thus $\lim_{a\to+\infty}\int_{\gamma_2} f(z)\mathrm{d}z = 0$ because e^{-a^2} converges to zero whereas the integral is bounded for any a.

Similarily

$$\int_{\gamma_4} f(z)\mathrm{d}z = -i\int_0^b e^{-(-a+i(b-t))^2}\mathrm{d}t = -ie^{-a^2}\int_0^b e^{2ai(b-t)+(b-t)^2}\mathrm{d}t$$

and thus as previously $\lim_{a\to+\infty}\int_{\gamma_4} f(z)\mathrm{d}z = 0$.

(iii) Using the parametrization introduced in the previous question, we find

$$\int_{\gamma_1} f(z)\mathrm{d}z = \int_{-a}^{a} e^{-t^2}\mathrm{d}t$$

and therefore

$$\lim_{a\to+\infty}\int_{\gamma_1} f(z)dz = \int_{-\infty}^{\infty} e^{-t^2}dt = \sqrt{\pi}.$$

On the other hand

$$\begin{aligned}\int_{\gamma_3} f(z)\mathrm{d}z &= -\int_{-a}^{a} e^{-(-t+bi)^2}\mathrm{d}t = -\int_{-a}^{a} e^{-t^2+2bti+b^2}dt\\ &= -e^{b^2}\int_{-a}^{a} e^{-t^2}(\cos(2bt)+i\sin(2bt))\mathrm{d}t,\end{aligned}$$

which implies that

$$\lim_{a\to+\infty}\int_{\gamma_3} f(z)dz = -e^{b^2}\int_{-\infty}^{\infty} e^{-t^2}\cos(2bt)\mathrm{d}t - ie^{b^2}\int_{-\infty}^{\infty} e^{-t^2}\sin(2bt)\mathrm{d}t.$$

Finally note that for every a we have $\int_\gamma f(z)\mathrm{d}z = 0$ and so letting $a\to+\infty$ we find

$$\begin{aligned}0 &= \int_\gamma f(z)\mathrm{d}z\\ &= \lim_{a\to+\infty}\int_\gamma f(z)\mathrm{d}z\\ &= \underbrace{\lim_{a\to+\infty}\int_{\gamma_1} f(z)\mathrm{d}z}_{=\sqrt{\pi}} + \underbrace{\lim_{a\to+\infty}\int_{\gamma_2} f(z)\mathrm{d}z}_{=0} + \lim_{a\to+\infty}\int_{\gamma_3} f(z)\mathrm{d}z + \underbrace{\lim_{a\to+\infty}\int_{\gamma_4} f(z)\mathrm{d}z}_{=0}\\ &= \sqrt{\pi} - e^{b^2}\int_{-\infty}^{\infty} e^{-t^2}\cos(2bt)\mathrm{d}t - ie^{b^2}\int_{-\infty}^{\infty} e^{-t^2}\sin(2bt)\mathrm{d}t.\end{aligned}$$

By identification of the real and imaginary parts, from this we can conclude

$$\int_{-\infty}^{\infty} e^{-x^2}\cos(2bx)dx = \sqrt{\pi}e^{-b^2}, \quad \int_{-\infty}^{\infty} e^{-x^2}\sin(2bx)dx = 0.$$

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6. Complex numbers and fluid dynamics.

Complex analysis has been a tool in fluid dynamics for a long time. Let $\mathcal{D} \subseteq \mathbb{R}^2$ be an open set. A vector field $\vec{F} : \mathcal{D} \to \mathbb{R}^2$ represents the velocity field of a fluid flow. The flow is called irrotational if $\operatorname{curl} \vec{F} = 0$ and incompressible if $\operatorname{div} \vec{F} = 0$ over $\mathcal{O}$.

(a) Show that the vector field

$$\vec{F} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2, \qquad (x_1, x_2) \mapsto \left(\frac{x_1}{x_1^2 + x_2^2}, \frac{x_2}{x_1^2 + x_2^2} \right) \tag{3}$$

is irrotational and incompressible.

(b) Represent $\vec{F}$ by a complex function $f : \mathcal{O} \to \mathbb{C}$ for some open set $\mathcal{O} \subseteq \mathbb{C}$. Show that f is complex differentiable.

Answer.

(a) We first compute the partial derivatives of the components:

$$\begin{aligned}
\partial_{x_1} \frac{x_1}{x_1^2 + x_2^2} &= \frac{1}{x_1^2 + x_2^2} - \frac{2x_1^2}{(x_1^2 + x_2^2)^2}, \\
\partial_{x_2} \frac{x_2}{x_1^2 + x_2^2} &= \frac{1}{x_1^2 + x_2^2} - \frac{2x_2^2}{(x_1^2 + x_2^2)^2}, \\
\partial_{x_2} \frac{x_1}{x_1^2 + x_2^2} = \partial_{x_1} \frac{x_2}{x_1^2 + x_2^2} &= \frac{2x_1 x_2}{(x_1^2 + x_2^2)^2}.
\end{aligned}$$

Now, one easily sees that $\vec{F}$ must incompressible and irrotational.

(b) As seen on the previous exercise sheet, the function

$$f(z) := f(x + yi) = \left(\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right) \tag{4}$$

is complex differentiable over $\mathcal{O} := \mathbb{C} \setminus \{0\}$.

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7. Contour integration.

Compute the following contour integrals.

(a) $\int_\gamma (z^2 + 1) \mathrm{d}z$ where $\gamma = [1, 1 + i]$ (segment between 1 and $1 + i$).

(b) $\int_\gamma \operatorname{Re}(z^2) \mathrm{d}z$, where $\gamma = \{z \in \mathbb{C} : |z| = 1\}$ (unit circle in 0).

Answer.

(a) We parametrize the segment as $\gamma(t) = 1 + it$ for $t \in [0, 1]$. Then applying the contour integral formula we find

$$\int_\gamma \left(z^2 + 1\right) dz = \int_0^1 \left((it + 1)^2 + 1\right) idt = i \int_0^1 \left(-t^2 + 2it + 2\right) dt = \frac{5i}{3} - 1.$$

(b) We parametrize the segment as $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$. Then applying the contour integral formula we find

$$\int_\gamma \operatorname{Re}\left(z^2\right) dz = \int_0^{2\pi} \operatorname{Re}\left(e^{2it}\right) ie^{it} dt = i \int_0^{2\pi} \cos(2t)(\cos(t) + i\sin(t))dt = 0.$$

The last step can be justified in many ways, one of which is a graphical argument that can be formalized by observing

$$\begin{aligned}\int_0^{2\pi} \cos(2t)(\cos(t) + i\sin(t)) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\left(2\left(t + \frac{\pi}{2}\right)\right) \cos\left(t + \frac{\pi}{2}\right) \\ &+ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\left(2\left(t + \frac{3\pi}{2}\right)\right) \cos\left(t + \frac{3\pi}{2}\right) \\ &+ i\int_{-\pi}^{\pi} \cos(2(t+\pi)) \sin(t+\pi).\end{aligned}$$

and that the three integrands are odd functions.

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Extra. Understanding complex numbers, once more If $z = x+iy$ is a complex number, what is the geometric interpretation of iz? More generally, for any $\theta \in \mathbb{R}$, what is the geometric interpretation of $e^{\theta i}z$? Finally, intepret this in terms of matrices.

Answer. If we interpret the complex number z as a vector $(x, y) \in \mathbb{R}^2$, then we can also interpret $iz = ix - y$ as the vector $(-y, x) \in \mathbb{R}^2$. In other words, multiplication with i can be interpreted as counterclockwise rotation by $90°$.

More generally, if $z = |z|e^{\phi i}$ has argument ϕ, then $e^{\theta i} \cdot |z|e^{\phi i} = |z|e^{(\theta+\phi)i}$ has argument θ. In other words, multiplication with $e^{\theta i}$ can be interpreted as counterclockwise rotation by θ radians.

As seen on the previous exercise sheet, we can interpret any complex number

$$e^{\theta i} = \cos(\theta) + \sin(\theta)i$$

as a matrix

$$M(e^{\theta i}) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

As (probably) discussed in linear algebra, this is a rotation matrix: the matrix-vector product of $M(e^{\theta i})$ and $(x, y)^t$ rotates the vector $(x, y)^t$ by the angle radian θ. ■