MATH-207(d) Analysis IV
Exercise session 3

1. Holomorphic functions.
Show that the following functions are holomorphic on $\mathbb{C}$ and compute their derivative.

(a) $\cos(z) = \dfrac{e^{iz} + e^{-iz}}{2}$.

(b) $\cosh(z) = \dfrac{e^{z} + e^{-z}}{2}$.

(c) $\sinh(z) = \dfrac{e^{z} - e^{-z}}{2}$.

Answer. As linear combination of holomorphic function, the three functions are holomorphic on $\mathbb{C}$. To compute theirs derivatives, we first compute their real and imaginary parts u and v and then use the formula $f'(z) = u_x + iv_x$ (or also $f'(z) = v_y - iu_y$).

(a)

$$\begin{aligned}\cos z &= \frac{e^{iz} + e^{-iz}}{2} \\ &= \frac{e^{-y}e^{ix} + e^{y}e^{-ix}}{2} \\ &= \frac{e^{y} + e^{-y}}{2}\cos x - i\frac{e^{y} - e^{-y}}{2}\sin x \\ &= \cosh y \cos x - i \sinh y \sin x.\end{aligned}$$

From which we conclude

$$\begin{aligned}u_x &= -\cosh y \sin x = v_y, \\ u_y &= \sinh y \cos x = -v_x,\end{aligned}$$

and so

$$(\cos z)' = u_x + iv_x = -(\cosh y \sin x + i \sinh y \cos x) = -\sin z.$$

To see the last equality just do the same computations as we did for $\cos(z)$.

(b)

$$\begin{aligned}\cosh z &= \frac{e^{z} + e^{-z}}{2} \\ &= \frac{e^{x}e^{iy} + e^{-x}e^{-iy}}{2} \\ &= \cosh x \cos y + i \sinh x \sin y.\end{aligned}$$

From which we conclude

$$\begin{aligned}u_x &= \sinh x \cos y = v_y, \\ u_y &= -\cosh x \sin y = -v_x,\end{aligned}$$

and so

$$(\cosh z)' = u_x + iv_x = \sinh z.$$

(c)

$$\begin{aligned}\sinh z &= \frac{e^z - e^{-z}}{2}\\ &= \frac{e^x e^{iy} - e^{-x}e^{-iy}}{2}\\ &= \sinh x \cos y + i \cosh x \sin y.\end{aligned}$$

From which we conclude

$$\begin{aligned}&u_x = \cosh x \cos y = v_y\\ &u_y = -\sinh x \sin y = -v_x\\ &(\sinh z)' = u_x + iv_x = \cosh z\end{aligned}$$

and so

$$(\sinh z)' = u_x + iv_x = \cosh z.$$

■

2. Holomorphic functions and harmonic maps.
Let $z = x + iy$, for $x, y \in \mathbb{R}$ and consider $f(z) = u(x, y) + iv(x, y)$, with $u, v \in C^2$. Show that if f is holomorphic on an open subset $\Omega \subseteq \mathbb{C}$, then u and v are harmonic functions, i.e.

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \Delta v = 0$$

Answer. Since u and v are of class C^2, then $u_{xy} = u_{yx}$ and $v_{xy} = v_{yx}$. Since f is holomorphic then the Cauchy-Riemann equations are satisfied:

$$u_x = v_y, \quad u_y = -v_x.$$

If we differentiate them, we find:

$$u_{xx} = v_{yx}, \quad u_{yy} = -v_{xy}$$

From which we conclude

$$\Delta u = u_{xx} + u_{yy} = v_{yx} - v_{yx} = 0.$$

Differentiating the Cauchy-Riemann equation with respect to y analogously leads to

$$\Delta v = v_{xx} + v_{yy} = 0.$$

■

3. More Holomorphic functions
Let $z = x + iy$, for $x, y \in \mathbb{R}$ and consider the function

$$f(z) = e^z = e^x e^{iy}.$$

Show that this function is holomorphic on $\mathbb{C}$ and compute its complex derivative.

Answer. Let $z = x + iy$ for some $x, y \in \mathbb{R}$. Using Euler's formula, we can deduce that

$$f(z) = e^z = e^x e^{iy} = \cos(y)e^x + i\sin(y)e^x.$$

Consequently, introducing the real-valued functions

$$u(x, y) = \cos(y)e^x \quad \text{and} \quad v(x, y) = \sin(y)e^x,$$

we then have

$$f(z) = u(x, y) + iv(x, y),$$

with u and v as defined above.
Obviously, u and v are differentiable functions. Moreover, a direct calculation now yields that for all $x, y \in \mathbb{R}$

$$\begin{aligned} \partial_x u(x, y) &= \cos(y)e^x, \quad & \partial_y u(x, y) &= -\sin(y)e^x, \\ \partial_x v(x, y) &= \sin(y)e^x, \quad & \partial_y v(x, y) &= \cos(y)e^x, \end{aligned}$$

and therefore,

$$\partial_x u = \partial_y v \quad \text{and} \quad \partial_y u = -\partial_x v.$$

Consequently, the Cauchy-Riemann equations are satisfied for all $x, y \in \mathbb{R}$, and we conclude that $f(z) = e^z$ is holomorphic on $\mathbb{C}$. Finally, the complex derivative of f is given by

$$\begin{aligned} f'(z) &= \partial_x u(x, y) + i\partial_x v(x, y) \\ &= \cos(y)e^x + i\sin(y)e^x \\ &= e^{iy}e^x \\ &= e^z, \end{aligned}$$

where the second from last step follows from Euler's formula. ■

4. Verifying Complex Differentiability
Express each of the following functions in the form

$$f(z) = u(x, y) + iv(x, y),$$

where u and v are real-valued functions and we represent any $z \in \mathbb{C}$ as $z = x + iy$ for some $x, y \in \mathbb{R}$.

Verify that the Cauchy-Riemann equations hold and compute the complex derivative.

(a) $f(z) = z^4$ over $\mathbb{C}$.

(b) $f(z) = e^{2z}$ over $\mathbb{C}$.

(c) $f(z) = 2z^2 + 1$ over $\mathbb{C}$.

(d) $f(z) = \dfrac{1}{z}$ over $\mathbb{C} \setminus \{0\}$.

(e) $f(z) = \dfrac{1}{z^2}$ over $\mathbb{C} \setminus \{0\}$.

Answer. Throughout this solution, we will represent any $z \in \mathbb{C}$ as $z = x + iy$ for some $x, y \in \mathbb{R}$.

(a) A direct calculation (using, e.g., the binomial theorem) reveals that

$$f(z) = z^4 = (x+iy)^4 = \underbrace{x^4 - 6x^2y^2 + y^4}_{:=u(x,y)} + i\underbrace{\left(4x^3y - 4xy^3\right)}_{:=v(x,y)}.$$

Obviously, u and v are differentiable functions. Moreover, it is readily seen that for all $x, y \in \mathbb{R}$

$$\begin{aligned} \partial_x u(x,y) &= 4x^3 - 12xy^2, \quad \partial_y u(x,y) = -12x^2y + 4y^3, \\ \partial_x v(x,y) &= 12x^2y - 4y^3, \quad \partial_y v(x,y) = 4x^3 - 12xy^2, \end{aligned}$$

and therefore,

$$\partial_x u = \partial_y v \quad \text{and} \quad \partial_y u = -\partial_x v.$$

Consequently, the Cauchy-Riemann equations are satisfied for all $x, y \in \mathbb{R}$, and we conclude that f is holomorphic on $\mathbb{C}$. The complex derivative of f is given by

$$\begin{aligned} f'(z) &= \partial_x u(x,y) + i\partial_x v(x,y) \\ &= 4x^3 - 12xy^2 + i(12x^2y - 4y^3) \\ &= 4\left(x^3 - 3xy^2 + i3x^2y - iy^3\right) \\ &= 4(x+iy)^3 = 4z^3. \end{aligned}$$

(b) A direct calculation (see also the previous question) shows that

$$f(z) = e^{2z} = e^{2(x+iy)} = \underbrace{\cos(2y)e^{(2x)}}_{:=u(x,y)} + i\underbrace{\sin(2y)e^{2x}}_{:=v(x,y)}.$$

Obviously, u and v are differentiable functions. Moreover, it is readily seen that for all $x, y \in \mathbb{R}$

$$\begin{aligned} \partial_x u(x,y) &= 2\cos(2y)e^{2x}, \quad \partial_y u(x,y) = -2\sin(2y)e^{2x}, \\ \partial_x v(x,y) &= 2\sin(y)e^{2x}, \quad \partial_y v(x,y) = 2\cos(2y)e^{2x}, \end{aligned}$$

and therefore,

$$\partial_x u = \partial_y v \quad \text{and} \quad \partial_y u = -\partial_x v.$$

Consequently, the Cauchy-Riemann equations are satisfied for all $x, y \in \mathbb{R}$, and we conclude that f is holomorphic on $\mathbb{C}$. Finally, the complex derivative of f is given by

$$\begin{aligned} f'(z) &= \partial_x u(x,y) + i\partial_x v(x,y) \\ &= 2\cos(2y)e^{2x} + i2\sin(2y)e^{2x} \\ &= 2e^{2iy}e^{2x} \\ &= 2e^{2z}, \end{aligned}$$

where the second from last step follows from Euler's formula.

(c) A direct calculation reveals that

$$f(z) = 2z^2 + 1 = 2(x+iy)^2 + 1 = \underbrace{2x^2 - 2y^2 + 1}_{:=u(x,y)} + i\underbrace{(4xy)}_{:=v(x,y)}.$$

Obviously, u and v are differentiable functions. Moreover, it is readily seen that for all $x, y \in \mathbb{R}$

$$\begin{aligned} \partial_x u(x,y) &= 4x, \quad \partial_y u(x,y) = -4y, \\ \partial_x v(x,y) &= 4y, \quad \partial_y v(x,y) = 4x, \end{aligned}$$

and therefore,

$$\partial_x u = \partial_y v \quad \text{and} \quad \partial_y u = -\partial_x v.$$

Consequently, the Cauchy-Riemann equations are satisfied for all $x, y \in \mathbb{R}$, and we conclude that f is holomorphic on $\mathbb{C}$. The complex derivative of f is given by

$$\begin{aligned} f'(z) &= \partial_x u(x,y) + i\partial_x v(x,y) \\ &= 4x + i4y \\ &= 4(x+iy) \\ &= 4z. \end{aligned}$$

(d) Throughout this calculation, we assume that $z \in \mathbb{C} \setminus \{0\}$ so that if $z = x + iy$ then $(x,y) \neq (0,0)$. Under this assumption, we can deduce that

$$f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \underbrace{\frac{x}{x^2+y^2}}_{:=u(x,y)} + i\underbrace{\frac{-y}{x^2+y^2}}_{:=v(x,y)}.$$

Notice that u and v are differentiable functions on $\mathbb{R}^2 \setminus \{(0,0)\}$. Moreover, for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ we have that

$$\begin{aligned} \partial_x u(x,y) &= \frac{y^2 - x^2}{(x^2+y^2)^2}, \quad \partial_y u(x,y) = -2\frac{xy}{(x^2+y^2)^2}, \\ \partial_x v(x,y) &= 2\frac{xy}{(x^2+y^2)^2}, \quad \partial_y v(x,y) = \frac{y^2 - x^2}{(x^2+y^2)^2}, \end{aligned}$$

and therefore,

$$\partial_x u = \partial_y v \quad \text{and} \quad \partial_y u = -\partial_x v.$$

Consequently, the Cauchy-Riemann equations are satisfied for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, and we conclude that f is holomorphic on $\mathbb{C} \setminus \{0\}$. The complex derivative of f at any

$z \neq 0$ is given by

$$\begin{aligned} f'(z) &= \partial_x u(x,y) + i\partial_x v(x,y) \\ &= \frac{y^2 - x^2}{(x^2+y^2)^2} + i2\frac{xy}{(x^2+y^2)^2} \\ &= -\frac{x^2 - 2ixy - y^2}{(x^2+y^2)^2} \\ &= -\frac{(x-iy)^2}{(x+iy)^2(x-iy)^2} \\ &= -\frac{1}{(x+iy)^2} \\ &= -\frac{1}{z^2}. \end{aligned}$$

(e) Throughout this calculation, we once again assume that $z \in \mathbb{C} \setminus \{0\}$ so that if $z = x + iy$ then $(x,y) \neq (0,0)$. Using the results from the previous question, we first deduce that

$$f(z) = \frac{1}{z^2} = \frac{1}{(x+iy)^2} = \underbrace{\frac{x^2 - y^2}{(x^2+y^2)^2}}_{:=u(x,y)} + i\underbrace{\frac{-2xy}{(x^2+y^2)^2}}_{:=v(x,y)}.$$

We observe once again that u and v are differentiable functions on $\mathbb{R}^2 \setminus \{(0,0)\}$. Moreover, for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ we have that

$$\begin{aligned} \partial_x u(x,y) &= 2x\frac{3y^2 - x^2}{(x^2+y^2)^3}, \quad \partial_y u(x,y) = -2y\frac{3x^2 - y^2}{(x^2+y^2)^3}, \\ \partial_x v(x,y) &= 2y\frac{3x^2 - y^2}{(x^2+y^2)^3}, \quad \partial_y v(x,y) = 2x\frac{3y^2 - x^2}{(x^2+y^2)^3}, \end{aligned}$$

and therefore,

$$\partial_x u = \partial_y v \quad \text{and} \quad \partial_y u = -\partial_x v.$$

Consequently, the Cauchy-Riemann equations are satisfied for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, and we conclude that f is holomorphic on $\mathbb{C} \setminus \{0\}$. The complex derivative of f at any

$z \neq 0$ is given by

$$
\begin{aligned}
f'(z) &= \partial_x u(x,y) + i\partial_x v(x,y) \\
&= 2x\frac{3y^2 - x^2}{(x^2+y^2)^3} + i2y\frac{3x^2 - y^2}{(x^2+y^2)^3} \\
&= \frac{-2x^3 + 6xy^2 + i6x^2y - 2iy^3}{(x^2+y^2)^3} \\
&= -2\frac{x^3 - 3xy^2 - i3x^2y + iy^3}{(x^2+y^2)^3} \\
&= -2\frac{(x-iy)^3}{(x+iy)^3(x-iy)^3} \\
&= -2\frac{1}{(x+iy)^3} \\
&= -2\frac{1}{z^3}.
\end{aligned}
$$

■

5. The complex logarithm.
Let $z = x+iy$, for $x, y \in \mathbb{R}$ and denote $|z| = \sqrt{x^2+y^2}$ the modulus of z and $\arg z$ its argument. The (complex) logarithm of z is defined as

$$\log(z) = \log|z| + i\arg z, \tag{1}$$

where $-\pi < \arg z \leq \pi$ and $\log|z|$ is the natural logarithm of the real number $|z|$. Show that:

(a) The complex logarithm is well defined on $\mathbb{C} \setminus \{0\}$.

(b) The complex logarithm is holomorphic on

$$\mathcal{O} = \mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Im} z = 0 \text{ and } \operatorname{Re} z \leq 0\}$$

and the derivative is $\frac{\mathrm{d}\log(z)}{\mathrm{d}z} = \frac{1}{z}$, $\forall z \in \mathcal{O}$. *You are only expected to show this when the real part is positive, that is,* $\Re z > 0$

Answer.

(a) Let us look at the definition of $\arg z$:

$$\arg z = \begin{cases} \operatorname{arctg}\frac{y}{x} & x > 0, y \in \mathbb{R} \\ \frac{\pi}{2} & x = 0, y > 0 \\ -\frac{\pi}{2} & x = 0, y < 0 \\ \pi + \operatorname{arctg}\frac{y}{x} & x < 0, y \geq 0 \\ -\pi + \operatorname{arctg}\frac{y}{x} & x < 0, y < 0 \end{cases}$$

Since $\arg z$ is not defined if $(\operatorname{Re} z, \operatorname{Im} z) = (0,0)$, it is clear from its definition that $\log z$ is not defined in $z = 0$.

(b) To understand why we have to remove negative real numbers consider

$$\log(c+it) = \log\sqrt{c^2+t^2} + i\arg(c+it), c\in\mathbb{R}, c<0$$

In order to see that $\log z$ is not continuous in c if $c<0$, we will look at the one-sided limits of $\log(c+it)$ in $t=0$ for $c<0$. We have

$$\lim_{t\to 0^+}\log(c+it) = \log|c| + i\pi,$$
$$\lim_{t\to 0^-}\log(c+it) = \log|c| - i\pi.$$

Since these values are different, $\log z$ is not continuous for $\operatorname{Im} z = 0$ and $\operatorname{Re} z < 0$, and therefore cannot be holomorphic in that region.

Now, we can prove that $\log z$ is holomorphic on $\mathcal{O}$, that is that derivation exists for every $z\in\mathcal{O}$. We will use $\arg z = \operatorname{arctg}\frac{y}{x}$. Then we have

$$\log z = \log\sqrt{x^2+y^2} + i\operatorname{arctg}\frac{y}{x} = u(x,y) + iv(x,y)$$

It is clear that $u, v\in C^1(\mathcal{O})$, in fact:

$$\begin{array}{ll} u_x = \frac{x}{x^2+y^2}, & u_y = \frac{y}{x^2+y^2} \\ v_x = \frac{-y}{x^2+y^2} & v_y = \frac{x}{x^2+y^2}. \end{array}$$

From this we observe the Cauchy-Riemann equations are satisfied and we have

$$(\log z)' = u_x + iv_x = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{1}{z}.$$

Hence, $\log z$ is holomorphic on O.

. ■

6. The complex power.
Let $\gamma\in\mathbb{C}$ and define $f(z) = z^\gamma = e^{\gamma\log(z)}$.

(a) Show that in general f is holomorphic on $\mathcal{O} = \mathbb{C}\setminus\{z\in\mathbb{C} : \operatorname{Im} z = 0 \text{ and } \operatorname{Re} z \le 0\}$ and the derivative is $f'(z) = \gamma z^{\gamma-1}$.

(b) What can we say about the case $\gamma\in\mathbb{N}$?

Answer.

(a) This is a composition of power function and logarithm. Since logarithm is holomorphic on $\mathcal{O}$, and power function is holomorphic on $\mathbb{C}$ we can conclude that f is holomorphic on $\mathcal{O}$. Using the composition rule for derivation we get

$$f'(z) = e^{\gamma\log z}\cdot\gamma z^{-1} = \gamma z^{\gamma-1}$$

(b) Now, let $\gamma \in \mathbb{N}$. Consider $\gamma = 1$, i.e. $f(z) = z$. By the definition we have

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z - z_0}{z - z_0} = 1.$$

Hence, the derivation exists for all $z_0 \in \mathbb{C}$, and we can conclude that $f(z) = z$ is holomorphic on $\mathbb{C}$. Similarly, for $\gamma = 2$, i.e. $f(z) = z^2$ we have

$$f'(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0} = 2z_0$$

Hence, the derivation exists for all $z_0 \in \mathbb{C}$ and $f(z) = z^2$ is holomorphic on C. Using the product rule for derivation, by induction, we can conclude that $f(z) = z^\gamma, \gamma \in \mathbb{N}$ is holomorphic on $\mathbb{C}$.

■

7. Divergence and curl.
Suppose that $f(z) = u(x, y) + v(x, y)i$ is holomorphic. We define the vector field

$$F(x, y) = \begin{pmatrix} v \\ u \end{pmatrix}.$$

What is the divergence and the curl of F?

Answer. Recall that u and v satisfy the Cauchy-Riemann equations:

$$\partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v.$$

We use these equations:

$$\operatorname{div} \begin{pmatrix} v \\ u \end{pmatrix} = \partial_x v + \partial_y u = 0, \quad \operatorname{curl} \begin{pmatrix} v \\ u \end{pmatrix} = \partial_x u - \partial_y v = 0.$$

NB: There is an interpretation in terms of fluid dynamics. The vector field F describes a flow that is incompressible (zero divergence) and irrotational (zero curl). In the area of fluid dynamics, such flows are called *potential flows*. In other words, the Cauchy-Riemann equations (outside of pure mathematics) describe a special situation of a flowing fluid. ■