MATH-207(d) Analysis IV
Exercise session 2

1. Cartesian to polar representation. Find the absolute value of the following complex numbers in polar form, identifying both absolute value and principal argument:

$$z_1 = 3 + 4i, \qquad z_2 = -1 + i\sqrt{3}, \qquad z_3 = -2 - 2i, \qquad z_4 = 5i,$$
$$z_5 = -3, \qquad z_6 = 1 + i, \qquad z_7 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \qquad z_8 = -2i.$$

Hint: in computing the principal argument, first think of how you can obtain the angle from (x, y).

Answer. To express each complex number in polar form, we compute its absolute value and principal argument. The absolute value of a complex number $z = x + yi$ is given by:

$$|z| = \sqrt{x^2 + y^2}.$$

The principal argument θ is generally found using:

$$\theta = \arctan\left(\frac{y}{x}\right),$$

with quadrant considerations to determine the correct angle. This leads to:

$$|z_1| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5, \quad \theta_1 = \arctan\left(\frac{4}{3}\right) \approx 0.93 \text{ rad},$$
$$\Rightarrow z_1 = 5e^{i\theta_1}.$$

$$|z_2| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1 + 3} = \sqrt{4} = 2, \quad \theta_2 = \arctan(-\sqrt{3}) = \frac{2\pi}{3},$$
$$\Rightarrow z_2 = 2e^{i\frac{2\pi}{3}}.$$

$$|z_3| = \sqrt{(-2)^2 + (-2)^2} = \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2}, \quad \theta_3 = \arctan\left(\frac{-2}{-2}\right) = \frac{5\pi}{4},$$
$$\Rightarrow z_3 = 2\sqrt{2}e^{i\frac{5\pi}{4}}.$$

$$|z_4| = \sqrt{0^2 + 5^2} = 5, \quad \theta_4 = \frac{\pi}{2},$$
$$\Rightarrow z_4 = 5e^{i\frac{\pi}{2}}.$$

$$|z_5| = \sqrt{(-3)^2 + 0^2} = 3, \quad \theta_5 = \pi,$$
$$\Rightarrow z_5 = 3e^{i\pi}.$$

$$|z_6| = \sqrt{1^2 + 1^2} = \sqrt{2}, \quad \theta_6 = \arctan(1) = \frac{\pi}{4},$$
$$\Rightarrow z_6 = \sqrt{2}e^{i\frac{\pi}{4}}.$$

$$|z_7| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1, \quad \theta_7 = \frac{2\pi}{3},$$
$$\Rightarrow z_7 = e^{i\frac{2\pi}{3}}.$$

$$|z_8| = \sqrt{0^2 + (-2)^2} = 2, \quad \theta_8 = \frac{3\pi}{2},$$
$$\Rightarrow z_8 = 2e^{i\frac{3\pi}{2}}.$$

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2. Polar to Cartesian representation. Find the Cartesian representation of the following complex numbers:

(a) $r = 2$, $\theta = \frac{\pi}{4}$

(b) $r = 3$, $\theta = -\frac{\pi}{3}$

(c) $r = 4$, $\theta = \pi$

(d) $r = 1$, $\theta = \frac{3\pi}{2}$

Answer. We use the conversion formula, as seen in the lecture:

$$z = re^{i\theta} = r\cos\theta + ir\sin\theta.$$

Applying this formula to each case:

(a) $r = 2$, $\theta = \frac{\pi}{4}$

$$\begin{aligned} z = 2e^{i\frac{\pi}{4}} &= 2\cos\frac{\pi}{4} + i2\sin\frac{\pi}{4} \\ &= 2 \cdot \frac{\sqrt{2}}{2} + i2 \cdot \frac{\sqrt{2}}{2} \\ &= \sqrt{2} + i\sqrt{2}. \end{aligned}$$

(b) $r = 3$, $\theta = -\frac{\pi}{3}$

$$\begin{aligned} z = 3e^{-i\frac{\pi}{3}} &= 3\cos\left(-\frac{\pi}{3}\right) + i3\sin\left(-\frac{\pi}{3}\right) \\ &= 3 \cdot \frac{1}{2} + i3 \cdot \left(-\frac{\sqrt{3}}{2}\right) \\ &= \frac{3}{2} - i\frac{3\sqrt{3}}{2}. \end{aligned}$$

(c) $r = 4$, $\theta = \pi$

$$\begin{aligned} z &= 4e^{i\pi} = 4\cos\pi + i4\sin\pi \\ &= 4 \cdot (-1) + i4 \cdot 0 \\ &= -4. \end{aligned}$$

(d) $r = 1$, $\theta = \frac{3\pi}{2}$

$$\begin{aligned} z &= 1e^{i\frac{3\pi}{2}} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} \\ &= 0 + i(-1) \\ &= -i. \end{aligned}$$

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3. Representation of functions. Write $f : \mathbb{C} \to \mathbb{C}$ in the form $f = u + iv$, where u and v are its real and imaginary parts.

- $f(z) = z^3$
- $f(z) = \frac{1}{z+1}$
- $f(z) = e^{2z}$
- $f(z) = \sin z$
- $f(z) = \cos z$
- $f(z) = \sinh z$
- $f(z) = \cosh z$
- $f(z) = \frac{z}{z+1}$
- $f(z) = \frac{1}{z^2+1}$
- $f(z) = \frac{z^2}{z-i}$

Determine a formula for the absolute value in each case, simplifying as much as possible.

Answer. To express $f : \mathbb{C} \to \mathbb{C}$ in the form $f = u + iv$, where u and v are the real and imaginary parts of f, we let $z = x + iy$ with $x, y \in \mathbb{R}$ and compute each function accordingly.

- $f(z) = z^3$
  Let $z = x + iy$, then:

  $$f(z) = (x + iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3).$$

  Hence,

  $$u(x, y) = x^3 - 3xy^2, \quad v(x, y) = 3x^2y - y^3.$$

  The absolute value is:

  $$|f(z)| = \sqrt{(x^3 - 3xy^2)^2 + (3x^2y - y^3)^2}.$$

- $f(z) = \frac{1}{z+1}$
  Writing $z = x + iy$, we get:
$$f(z) = \frac{1}{(x+1)+iy} \cdot \frac{(x+1)-iy}{(x+1)-iy} = \frac{x+1-iy}{(x+1)^2+y^2}.$$
  Thus,
$$u(x,y) = \frac{x+1}{(x+1)^2+y^2}, \quad v(x,y) = \frac{-y}{(x+1)^2+y^2}.$$
  The absolute value is:
$$|f(z)| = \frac{1}{\sqrt{(x+1)^2+y^2}}.$$

- $f(z) = e^{2z}$
$$e^{2(x+iy)} = e^{2x}e^{i2y} = e^{2x}(\cos 2y + i \sin 2y).$$
  Hence,
$$u(x,y) = e^{2x}\cos 2y, \quad v(x,y) = e^{2x}\sin 2y.$$
  The absolute value is:
$$|f(z)| = e^{2x}.$$

- $f(z) = \sin z$
  Using $\sin z = \sin x \cosh y + i \cos x \sinh y$,
$$u(x,y) = \sin x \cosh y, \quad v(x,y) = \cos x \sinh y.$$
  The absolute value is:
$$|f(z)| = \sqrt{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y}.$$

- $f(z) = \cos z$
  Using $\cos z = \cos x \cosh y - i \sin x \sinh y$,
$$u(x,y) = \cos x \cosh y, \quad v(x,y) = -\sin x \sinh y.$$
  The absolute value is:
$$|f(z)| = \sqrt{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}.$$

- $f(z) = \sinh z$
  Using $\sinh z = \sinh x \cos y + i \cosh x \sin y$,
$$u(x,y) = \sinh x \cos y, \quad v(x,y) = \cosh x \sin y.$$
  The absolute value is:
$$|f(z)| = \sqrt{\sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y}.$$

- $f(z) = \cosh z$
  Using $\cosh z = \cosh x \cos y + i \sinh x \sin y$,
$$u(x,y) = \cosh x \cos y, \quad v(x,y) = \sinh x \sin y.$$
  The absolute value is:
$$|f(z)| = \sqrt{\cosh^2 x \cos^2 y + \sinh^2 x \sin^2 y}.$$

- $f(z) = \frac{z}{z+1}$

$$\begin{aligned} f(z) &= \frac{x+iy}{(x+1)+iy} \cdot \frac{(x+1)-iy}{(x+1)-iy} = \frac{(x+iy)(x+1-iy)}{(x+1)^2+y^2} \\ &= \frac{x^2+ixy+x+iy-ixy+y^2}{(x+1)^2+y^2} \\ &= \frac{x^2+x+iy+y^2}{(x+1)^2+y^2}. \end{aligned}$$

Simplifying,

$$u(x,y) = \frac{x(x+1)+y^2}{(x+1)^2+y^2}, \quad v(x,y) = \frac{y}{(x+1)^2+y^2}.$$

The absolute value is:

$$|f(z)| = \frac{\sqrt{x^2+y^2}}{\sqrt{(x+1)^2+y^2}}.$$

- $f(z) = \frac{1}{z^2+1}$
Expanding $(x+iy)^2+1$,

$$f(z) = \frac{1}{(x^2-y^2+1)+i(2xy)} = \frac{x^2-y^2+1-i2xy}{(x^2-y^2+1)^2+4x^2y^2}.$$

Thus,

$$u(x,y) = \frac{x^2-y^2+1}{(x^2-y^2+1)^2+4x^2y^2}, \quad v(x,y) = \frac{-2xy}{(x^2-y^2+1)^2+4x^2y^2}.$$

The absolute value is:

$$|f(z)| = \frac{1}{\sqrt{(x^2-y^2+1)^2+4x^2y^2}}.$$

- $f(z) = \frac{z^2}{z-i}$

$$\begin{aligned} f(z) &= \frac{(x+iy)^2}{x+i(y-1)} = \frac{(x+iy)^2(x-i(y-1))}{x^2+(y-1)^2} \\ &= \frac{(x^2-y^2+2ixy)(x-iy+i)}{x^2+(y-1)^2} \\ &= \frac{x^3-xy^2+2ix^2y-x^2yi+y^3i+2xy^2+x^2i-y^2i-2xy}{x^2+(y-1)^2} \\ &= \frac{x^3+ix^2y+y^3i+xy^2+x^2i-y^2i-2xy}{x^2+(y-1)^2} \\ &= \frac{x^3+xy^2-2xy+(x^2y+y^3+x^2-y^2)i}{x^2+(y-1)^2}. \end{aligned}$$

Expanding and simplifying,

$$u(x,y) = \frac{x^3+xy^2-2xy}{x^2+(y-1)^2}, \quad v(x,y) = \frac{x^2y+y^3+x^2-y^2}{x^2+(y-1)^2}.$$

The absolute value is:

$$|f(z)| = \frac{x^2+y^2}{\sqrt{x^2+(y-1)^2}}.$$

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4. Double exponentials.
Determine the real and imaginary part of the following functions:

$$f(z) = e^{e^z}, \quad g(z) = \frac{1}{f(z)},$$

where $z = x + iy$. Describe the function f in the two special cases $x = 0$ and $y = 0$.

Answer. Using the definition of the complex exponential, we find

$$\begin{aligned} f(z) &= e^{e^z} \\ &= e^{e^x \cos(y) + e^x \sin(y) i} \\ &= e^{e^x \cos(y)} e^{e^x \sin(y) i} \\ &= e^{e^x \cos(y)} \Big(\cos\left(e^x \sin(y)\right) + \sin\left(e^x \sin(y)\right) i \Big) \\ &= e^{e^x \cos(y)} \cos\left(e^x \sin(y)\right) + e^{e^x \cos(y)} \sin\left(e^x \sin(y)\right) i. \end{aligned}$$

We obtain the formula for g similarly. We can also observe that $|f(z)| = e^{e^x \cos(y)}$ and immediately conclude

$$g(z) = e^{-e^x \cos(y)} \cos\left(e^x \sin(y)\right) - e^{-e^x \cos(y)} \sin\left(e^x \sin(y)\right) i.$$

In the special case $y = 0$, this formula just gives back the double exponential e^{e^x} in the one single variable x. In the special case $x = 0$, we get

$$f(z) = f(iy) = e^{\cos(y)} \cos\left(\sin(y)\right) + e^{\cos(y)} \sin\left(\sin(y)\right) i.$$

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5. Matrix representation of complex numbers.
Given a complex number $z = x + yi$, we define a matrix

$$M(z) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

Show the following:

$$M(z_1 + z_2) = M(z_1) + M(z_2), \quad M(z_1 \cdot z_2) = M(z_1) \cdot M(z_2),$$
$$M(z)^{-1} = M\left(z^{-1}\right) \text{ if } z \neq 0.$$

Answer. We write $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$. We easily obverse

$$M(z_1 + z_2) = \begin{pmatrix} x_1 + x_2 & -y_1 - y_2 \\ y_1 + y_2 & x_1 + x_2 \end{pmatrix} = \begin{pmatrix} x_1 & -y_1 \\ -x_1 & y_1 \end{pmatrix} + \begin{pmatrix} x_2 & -y_2 \\ -x_2 & y_2 \end{pmatrix} = M(z_1) + M(z_2).$$

We also check that

$$M(z_1) \cdot M(z_2) = \begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{pmatrix} = \begin{pmatrix} x_1 x_2 - y_1 y_2 & -x_1 y_2 - x_2 y_1 \\ x_1 y_2 + x_2 y_1 & x_1 x_2 - y_1 y_2 \end{pmatrix} = M(z_1 \cdot z_2).$$

Finally, if $z = x + yi$ is not zero, then

$$M(z)^{-1} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}^{-1} = \frac{1}{x^2 - y^2} \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$
$$M\left(z^{-1}\right) = M\left(\frac{\bar{z}}{z \cdot \bar{z}}\right) = M\left(\frac{x - yi}{x^2 + y^2}\right) = \begin{pmatrix} \frac{x}{x^2 - y^2} & \frac{y}{x^2 - y^2} \\ \frac{-y}{x^2 - y^2} & \frac{x}{x^2 - y^2} \end{pmatrix}.$$

NB: Any such matrix $M := M(z)$ satisfies $M_{11} = M_{22}$ and $M_{21} = -M_{12}$. ■

6. Review Fourier series and Poisson problem.
Consider the boundary value problem of finding a function $u : [0,1] \to \mathbb{R}$ such that

$$\begin{aligned} u''(x) &= x, \qquad 0 < x < 1, \\ u(0) &= u(1) = 0. \end{aligned}$$

Express the solution as a Fourier series.

Answer. We explore two possible solutions: either we solve the boundary value problem manually and compute its Fourier series, or we first compute the Fourier series and from there the Fourier series of the solution.
1. First approach To compute the Fourier series of the right-hand side (RHS), also known as source term, $f(x) = x$ and derive the solution $u(x)$, we proceed as follows. We expand $f(x) = x$ as a **Fourier sine series** on the interval $[0,1]$,

$$x = \sum_{n=1}^{\infty} f_n \sin(n\pi x),$$

where the coefficients b_n satisfy the formula

$$f_n = 2\int_0^1 x \sin(n\pi x)\, dx.$$

A simple integration by parts provides

$$2\int_0^1 x\sin(n\pi x)\,dx = -2\frac{x\cos(n\pi x)}{n\pi}\bigg|_0^1 + 2\frac{1}{n\pi}\int_0^1 \cos(n\pi x)\,dx = 2\cdot\frac{(-1)^{n+1}}{n\pi} = \frac{2(-1)^{n+1}}{n\pi}.$$

The function $f(x) = x$ can thus be written

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi}\sin(n\pi x), \qquad 0 < x < 1.$$

We assume that $u(x)$ has the Fourier sine series of the form

$$u(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x).$$

Differentiating this twice produces

$$u''(x) = -\sum_{n=1}^{\infty} b_n (n\pi)^2 \sin(n\pi x).$$

Equating the coefficients of $u''(x)$ and $f(x)$, we finally obtain the coefficients

$$\begin{aligned} -\sum_{n=1}^{\infty} b_n(n\pi)^2\sin(n\pi x) &= \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n\pi}\sin(n\pi x). \\ \implies -b_n(n\pi)^2 &= \frac{2(-1)^{n+1}}{n\pi} \\ \implies b_n &= \frac{2(-1)^n}{(n\pi)^3}. \end{aligned}$$

This result coincides with the one obtained by directly solving the boundary value problem and expanding the solution into a Fourier series, to be discussed next.
2. Second approach To begin with, we integrate $u''(x) = x$ twice,

$$u'(x) = \frac{1}{2}x^2 + C, \qquad u(x) = \frac{1}{6}x^3 + Cx + D,$$

and apply the boundary conditions $u(0) = 0$ and $u(1) = 0$ to determine the coefficients:

$$u(0) = 0 \implies D = 0$$
$$u(1) = 0 \implies \frac{1}{6} + C = 0 \implies C = -\frac{1}{6}.$$

We conclude that the solution has the form

$$u(x) = \frac{1}{6}x^3 - \frac{1}{6}x = \frac{x^3 - x}{6}.$$

From here, we compute the Fourier series expansion, writing $u(x) = \frac{x^3-x}{6}$ as

$$u(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

with unknown coefficients b_n. We must calculate

$$b_n = 2\int_0^1 \frac{x^3 - x}{6} \sin(n\pi x)\, dx = \frac{1}{3}\int_0^1 (x^3 - x)\sin(n\pi x)\, dx.$$

Some simple computation, via integration by parts, shows

$$b_n = \frac{1}{3}\left(\int_0^1 x^3 \sin(n\pi x)\, dx - \int_0^1 x \sin(n\pi x)\, dx\right) = \frac{1}{3} \cdot \frac{6(-1)^n}{(n\pi)^3} = \frac{2(-1)^n}{(n\pi)^3}.$$

Thus we arrive at the Fourier series of the solution

$$u(x) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{(n\pi)^3} \sin(n\pi x),$$

in agreement with our first approach to this problem. ■