Review topics – Cauchy Problem

1. order Cauchy problem: $u' + cu = 0, \quad u(0) = u_0$

Laplace transform, isolate $U(z)$, inverse Laplace transform

$zU(z) - u(0) + cU(z) = 0 \quad \Rightarrow \quad U(z) = u_0 \frac{1}{z+c} \quad \Rightarrow \quad u(t) = u_0 e^{-ct}$

Now, with source term: $u' + cu = f, \quad u(0) = u_0$

$$u(t) = u_0 e^{-ct} + \int_0^t f(s) e^{-c(t-s)} ds$$

2. order Cauchy problem $u'' + 2\alpha u' + \omega^2 u = 0, \quad u(0) = u_0, \quad u'(0) = v_0$

Laplace transform, isolate $U(z)$, inverse Laplace transform

Case $\omega^2 - \alpha^2 = 0$

$$u(t) = e^{-\alpha t}\left(u_0 + t(\alpha u_0 + v_0) \right)$$

Case $\omega^2 - \alpha^2 > 0$

$$u(t) = u_0 e^{-\alpha t} \cos\left(\sqrt{\omega^2 - \alpha^2}\, t \right) + \frac{\alpha u_0 + v_0}{\sqrt{\omega^2 - \alpha^2}} e^{-\alpha t} \sin\left(\sqrt{\omega^2 - \alpha^2}\, t \right)$$

Case $\omega^2 - \alpha^2 < 0$

$$u(t) = u_0 e^{-\alpha t} \cosh\left(\sqrt{\alpha^2 - \omega^2}\, t \right) + \frac{\alpha u_0 + v_0}{\sqrt{\alpha^2 - \omega^2}} e^{-\alpha t} \sinh\left(\sqrt{\alpha^2 - \omega^2}\, t \right)$$

Solution to Poisson problem: $-u'' + k^2 u = 0$

$$u(t) = u(0)\cosh(kt) + \frac{u'(0)}{k}\sinh(kt)$$

If we want $u(0) = g_a$ and $u(b) = g_b$, then we set

$$u'(0) = \frac{k}{\sinh(kb)}\left(g_b - g_a \right)$$

Review topics – Partial Differential Eq.

Important partial differential equations, appearing in many different variants

Heat Equation $\partial_t u(x,t) = \partial_{xx} u(x,t) + f(x,t), \quad 0<x<L, \quad t>0$

Boundary conditions $\quad u(0,t) = 0, \quad u(L,t) = 0$

Initial conditions $\quad u(x,0) = u_0(x), \quad 0<x<L$

Schrödinger Equation $i\,\partial_t u(x,t) = \partial_{xx} u(x,t) + f(x,t), \quad 0<x<L, \quad t>0$

Boundary conditions $\quad u(0,t) = 0, \quad u(L,t) = 0$

Initial conditions $\quad u(x,0) = u_0(x), \quad 0<x<L$

Wave Equation $\partial_{tt} u(x,t) = \partial_{xx} u(x,t) + f(x,t), \quad 0<x<L, \quad t>0$

Boundary conditions $\quad u(0,t) = 0, \quad u(L,t) = 0$

Initial conditions $\quad u(x,0) = u_0(x), \quad 0<x<L$

$\partial_t u(x,0) = v_0(x), \quad 0<x<L$

Schema Fourier series

1. Write solution u, source term f, and initial values u_0 (v_0) in terms of Fourier sine series

2. Use partial differential equation to get Cauchy problem for each Fourier coefficient

Heat: $\quad b_n'(t) = -\left(\frac{\pi n}{L}\right)^2 b_n + b_n^f(t), \quad b_n(0) = b_n^0$

Schrödinger: $\quad i\, b_n'(t) = -\left(\frac{\pi n}{L}\right)^2 b_n + b_n^f(t), \quad b_n(0) = b_n^0$

Wave: $\quad b_n''(t) = -\left(\frac{\pi n}{L}\right)^2 b_n + b_n^f(t), \quad b_n(0) = b_n^0, \; b_n'(0) = v_n^0$

3. Solve Cauchy problem to get $b_n(t)$. Full solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{\pi n}{L} x\right)$$

Schéma Fourier transform

1. Find Fourier transforms of solution u, source term f, and initial values u_0 (v_0)

2. Take Fourier transform of partial differential equation

Heat: $\partial_t \hat{u}(\alpha,t) = -\alpha^2 \hat{u}(\alpha,t) + \hat{f}(\alpha,t)$

Schrödinger: $i\partial_t \hat{u}(\alpha,t) = -\alpha^2 \hat{u}(\alpha,t) + \hat{f}(\alpha,t)$

Wave: $\partial_{tt} \hat{u}(\alpha,t) = -\alpha^2 \hat{u}(\alpha,t) + \hat{f}(\alpha,t)$

3. For each frequency term $\alpha \in \mathbb{R}$, solve Cauchy problem [needs FT of initial values]

4. Having found $\hat{u}(\alpha,t)$, apply inverse FT to get $u(x,t)$